VOLUME XXXIV NUMBER 3

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED
WITH THE CO-OPERATION OF THE

AMERICAN PHYSICAL SOCIETY

BY

EDWARD L. NICHOLS,
ERNEST MERRITT, and FREDERICK BEDELL

CLXXXIV

SEPTEMBER, 1911

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THE PHYSICAL REVIEW

LANCASTER, PA., AND ITHACA, N. Y.

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IN THE UNITED STATES

Annual Subscription $5.00 Single Copies $ .60

OUTSIDE THE UNITED STATES

Annual Subscription $5.50 Single Copies $ .65

Two volumes of THE PHYSICAL REVIEW are published annually, these volumes beginning in January and July, respectively, and containing six numbers each. Subscriptions should be sent to the publishers, THE PHYSICAL REVIEW, *41 North Queen Street, Lancaster, Pa.*, or *Ithaca, N. Y.*; or to Messrs. MAYER AND MÜLLER, *Berlin*. In order that subscriptions may terminate with the December issue, subscriptions for less than one year, terminating in December, will be taken at the yearly rate.

Previous to Volume V (July–December, 1897) THE PHYSICAL REVIEW was published in annual volumes, each containing six bi-monthly numbers, beginning with the July–August number, 1893.

A limited number of complete sets of the REVIEW will be supplied at the price of $3.00 per volume for all volumes, carriage extra. When they can be furnished without breaking a set, separate back volumes will be supplied at the same price, and single back numbers will be mailed to any address in the United States at the price of $.60 per copy, or to any address outside the United States at the price of $.65 per copy.

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Entered at the post-office at Lancaster, Pa., as second class matter.

Volume XXXIII. *September, 1911.* *Number 3*

THE

PHYSICAL REVIEW.

THE TERMINAL VELOCITY OF FALL OF SMALL SPHERES IN AIR AT REDUCED PRESSURES.

By L. W. McKeehan.

§ I. Introduction.

THE terminal velocity of a freely falling sphere, when all other resistances are negligible compared to that due to the viscosity of the fluid through which the sphere falls, and when no slipping is supposed to occur at the surface of separation, is, as derived by Stokes,[1]

$$V = \frac{\frac{2}{9}ga^2(\sigma - \rho)}{\mu}.$$

In this expression μ is the coefficient of viscosity of the fluid, a the radius of the sphere, V the velocity of fall, g the acceleration of gravity, σ the density of the sphere, and ρ that of the fluid. The exclusion of all but viscous resistance restricts the applicability of the formula to very minute spheres, the condition to be fulfilled being that the radius of the sphere must be small compared to $\mu/V\rho$.

Experimental work at atmospheric pressure by Professor John Zeleny and the writer[2] has shown this formula to hold for spheres of wax ranging in radius from .002 cm. to .00035 cm., spheres of paraffin from .002 cm. to .0005 cm., and spheres of mercury from .001 cm. to .00016 cm., although the measurements on the last named are less accurate than for the other materials, owing to the high reflecting power of their surfaces.

In earlier experiments[3] using natural spores as approximating small spheres, large deviations from the formula given by Stokes were observed, all the spores going too slowly to agree with the formula. It was sug-

[1] G. G. Stokes, Mathematical and Physical Papers, Vol. III., p. 59.

[2] Phys. Rev., Vol. XXX., p. 535; Phys. Zeitschr., Bd. 11, s. 78.

[3] Loc. cit.

gested by Sir Joseph Larmor at the Winnipeg Meeting of the British Association for the Advancement of Science, August, 1909, that experiments at lower pressures would serve to decide whether or not these deviations were due to the formation of eddies in the air near the not perfectly spherical spores. The force required to maintain these eddies would decrease with the density of the gas, whereas the purely viscous resistance should remain constant, since μ is independent of pressure, at least throughout a very great range.[1] If the large deviations observed with spores were due to this cause, the terminal velocity of fall at a lower pressure should be larger than at atmospheric pressure, and should tend to agree more closely with that given by the formula of Stokes.

Some experiments at low pressures, using lycopodium spores, were performed. These showed, indeed, an increase in terminal velocity, but one of far too great an amount to be due to the suspected cause, since the velocity became at low pressures several times as great as that calculated by the formula. Similar experiments were then performed with perfect spheres of wax, which satisfy Stokes's formula at atmospheric pressure. These showed likewise a large increase in terminal velocity with reduction of pressure, the data so obtained forming the experimental part of the paper here presented. This failure of Stokes's formula at low pressures is traceable to the omission of slip in his derivation. Such slipping of the gas along the surface of the sphere is easily shown to be inseparable from motion in a discontinuous medium, and becomes more important with reduction of pressure as the mean distance between the molecules increases.

The necessity of slip can be seen from the following considerations. The surface layer of molecules in a gas which is moving along a solid surface may be divided into two groups. The first group consists of those which have not yet struck the surface. Their mean velocity will have the component parallel to the surface of the more distant layer of gas from which they come. The second group consists of those molecules which have already struck. Their mean velocity will have a component parallel to the surface smaller than that of the first group, but greater than zero unless the impacts are entirely inelastic. In any case the mean velocity of both groups taken together, which is that of the gas layer as a whole, will have a tangential component, or, in other words, there will be slip between the gas and the solid surface.

[1] Weinstein, Thermodynamik und Kinetik der Körper, Bd. I., s. 325.

§ II. Theory.

A theoretical discussion of the case was given by Professor E. Cunningham[1] in a paper "On the Velocity of Steady Fall of Spherical Particles through Fluid Medium," which was communicated to the Royal Society of London in January, 1910, by Sir Joseph Larmor. He obtains as the general expression for the terminal velocity of fall of a small sphere in a gas

$$V = V_s\left(1 + A\frac{l}{a}\right),$$

in which V_s is the velocity given by Stokes's formula, l is the mean free path of the gas molecule, a the radius of the sphere, and A a constant whose numerical value depends on the assumption which is made regarding the impacts of the gas molecules on the surface of the sphere. An examination of the results there obtained has revealed some inaccuracies. When these are corrected different values of A than those he obtained are derived.

Suppose in the first instance that the collisions are of the nature of impacts of smooth elastic spheres; the value of A is then 1.5.[2] Suppose next that each impinging molecule enters the surface layer of the particle (whether the solid material or a layer of condensed gas is for the present purpose immaterial), and emerges again from the same area on which it impinged, but with a velocity independent in direction of the relative velocity of the sphere and the molecule before the collision, although of the same mean squared value; the value of A is then 1.2. Suppose that a fraction f of the impacts are of the former type and the rest of the latter; the value of A is then $1.5\frac{4}{5-f}$. These cover the cases treated by Cunningham, who obtained in the combined case $A = 1.63\frac{1}{2-f}$.

As an additional and limiting case suppose that after striking the surface of a particle a molecule of the gas is only able to leave it again when moving in the direction of the normal to the surface; the value of A is then 1.05.

Since the mean free path l of the gas molecule is inversely proportional to the pressure p of the gas, an alternative formula

$$V = V_s(1 + B/ap)$$

[1] Proc. Roy. Soc., Vol. 83 A, p. 357, February, 1910.

[2] One of the substitutions was made incorrectly in a note published in the Physical Review, Vol. XXXII., p. 341, March, 1911, so that the numerical value of the constant multiplier of l/a is there given wrongly. This mistake was pointed out by Professor R. A. Millikan, who kindly read a part of the analysis leading to the results here given.

can be used, and is more convenient for experimental verification. Calculating the value of B from that of A, using $\mu = 1833 \times 10^{-7}$ [1] at 20° C., we get when p is expressed in millimeters of mercury, for

$A = 1.0$	$B = .0075$
1.05	.0078
1.2	.0090
1.5	.0 12

The terminal velocity of fall of small spheres should thus increase enormously at very low pressures, and should then become proportional to the radius of the sphere, and not to the square of the radius as at higher pressures. It should also be inversely proportional to the pressure. In Figs. 1 and 2 are plotted as abscissa the reciprocal of the computed velocity, that is, the time required to fall one centimeter at this velocity, and as ordinate the pressure in mm. of mercury. The temperature

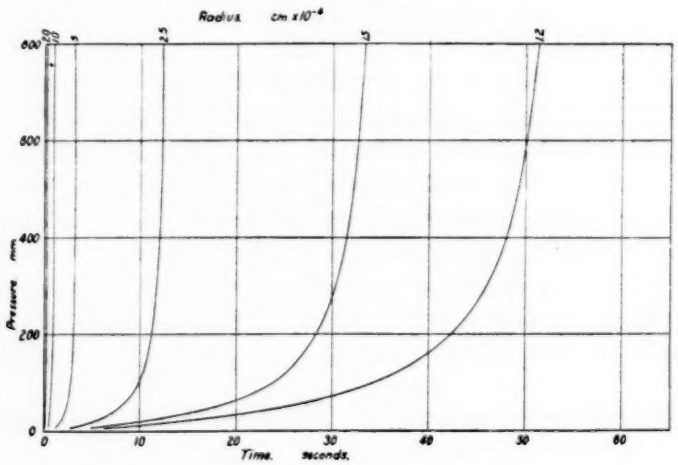

Fig. 1.

assumed is 20° C., and the value of the constant chosen is $A = 1.0$, since, as shown later, this is the value obtained by experiment. Six curves are drawn in Fig. 1 for different values of the radius, the value for each curve being given in cm. $\times 10^{-4}$ in the upper margin. Fig. 2 shows on a much larger scale the part near the origin omitted from Fig. 1. The close approach to a linear variation of time of fall with pressure should here be noted. It is found upon computation that the criterion for steady motion given above is fulfilled by all the sizes shown at all the pressures.

The assumption is made that μ has the same value at all pressures. This is required by the kinetic theory of gases and it seems likely that

[1] R. A. Millikan, Phil. Mag., VI., Vol. 19, p. 215, February, 1910, $\mu = 1863 \times 10^{-7}$ at 26° C.

the values of μ obtained experimentally at low pressures are somewhat lower than at atmospheric pressure only because slip is neglected. For this reason the pressure at which μ shows a marked decrease depends on

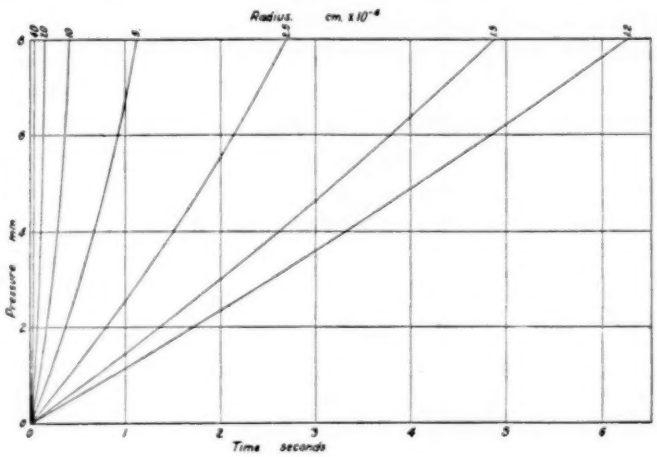

Fig. 2.

the method used in determining it, and upon the dimensions of the apparatus employed.

§ III. Experiments.

1. *Preparation of Material.*—Minute spheres of wax were made by an atomizer,[1] and were collected by allowing the cloud of spheres so formed to settle on sheets of paper in a closed space about 40 × 40 × 60 cm. The range of sizes obtained could be varied within certain limits by regulating the size and openings of the atomizer, and by varying the temperature of the wax.

2. *Measurements.*—In order to find the value of B in the formula

$$V = \frac{2}{9}\frac{ga^2(\sigma - \rho)}{\mu}\left(1 + \frac{B}{ap}\right),$$

the values of σ, the density of a sphere, a its radius, V its terminal velocity of fall, and p the pressure of the gas in millimeters of mercury, were determined for over six hundred spheres of wax.

3. *Density.*—The density σ of the wax was 1.058, as previously determined.[2]

4. *Terminal Velocity of Fall.*—The method of getting the terminal velocity of fall V was the same as that used in the previous experiments,

[1] The same method was used in the earlier experiments cited. Microphotographs of these perfect spheres are there shown.

[2] Loc. cit.

and consisted in measuring the total time required to fall a known distance of about 31 cm.

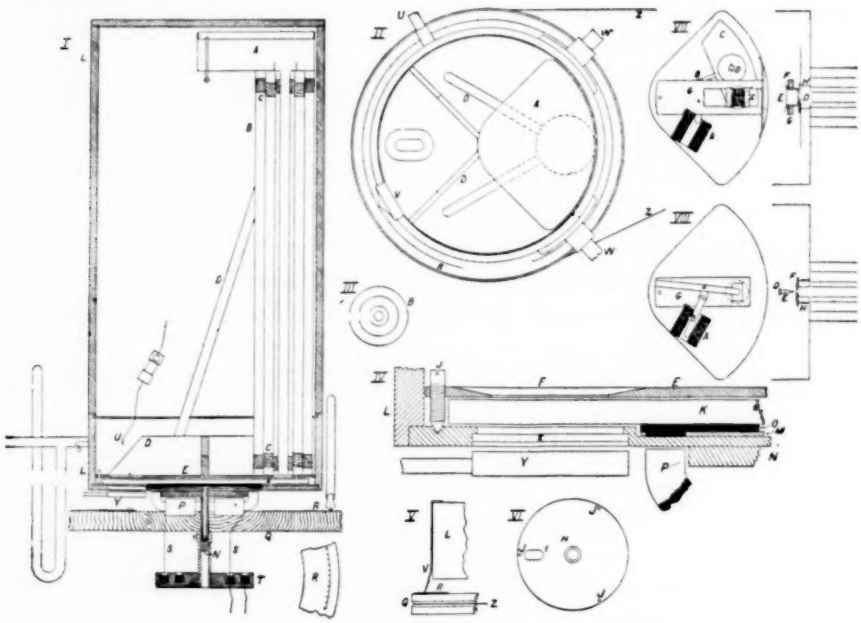

Fig. 3.

Referring to Fig. 3: drawing *I* is a vertical section through the apparatus; *II* is a plan of the same with the cover removed; *III* is a horizontal section through the tube in which the spheres fell; *IV* is an enlarged view of the lower left-hand corner of *I*; *V* and *VI* are details of parts not shown elsewhere; *VII* and *VIII* are plans and sections of two mechanisms used for releasing spheres. The lettering of corresponding parts is the same in the first six drawings.

The spheres released in the fall box *A* fell through the fall tube *B* to the revolving disc *K*. In releasing spheres by the method shown in drawing *VII*, a quantity of them several mm³. in volume was placed in the cloth-bottomed trough *E*, which slipped tightly into a slot in the frame *G*. A solenoid *A* drew in the soft-iron armature *B*, and by it moved the sliding plate *C* across the short open tube *H* soldered to the bottom of the fall box just under *E*. The sliding plate carried a flexible pointed spring *D* projecting above a small hole in its center. The motion of *C* was stopped by a peg, not shown, when *D* reached the middle of the bottom of the trough *E*, against which it scratched lightly. The vibration produced by this scratch sifted some spheres through the cloth and they then fell through the small hole in *C*, through *H*, and into the fall tube *B*.

In the method shown in drawing *VIII* the spheres were left on the paper upon which they had been deposited after atomizing. A piece of this was put at *F*, with the sphere-coated side underneath, just over a small hole in the frame *G*, which fit tightly in the tube *H*. The solenoid *A*, by means of the armature *B* drew a brass block *C* from beneath a long metal strip *D*, which sprung downward, rapping the point *E* against the upper surface of the paper just over the hole in *G*. The blow was not sufficient to puncture the paper, but it set some of the spheres free and they then fell through *H* into the fall tube.

In the fall tube *B* the spheres, released by either of the methods described, fell through the innermost of four concentric brass tubes, designed to prevent air currents. The outer tubes were pierced at one point, as shown in drawing *III*, to facilitate changes of pressure in the spaces between the wooden rings placed near the ends. The three inner tubes were entirely supported by these rings and were not in contact with any other metal parts. The apparatus was used in a room with double walls and no windows, which provided an excellently steady temperature.

The outer tube was soldered at the top of the fall box *A*, and at the bottom to braces *D*, *D*, and to the brass plate *E*. This plate was supported on three leveling screws *J* (drawing *IV*), one of which rested in a socket, fixing the relative angular position of the fall tube and the outer case *L*. The plate was pierced by two holes, one a narrow radial slot just under the fall tube, and the other a window, 180° from the slot, for viewing the spheres after they had fallen. A plate *H* (drawing *VI*) similar to *E*, but without any fall tube, was substituted for it when measuring the spheres. This provided more room for the measuring microscope.

The surface for receiving the spheres was that of a plate glass disc *K* cemented to a circular brass plate *M*, and rotated under the plate *E* on a pivot in the bearing *N*. On a diameter of the plate *M* was imbedded a soft-iron armature *O* with thickened ends. The poles of this armature were directly over the poles of an electromagnet *P* carried on a wooden wheel *Q*, which was rotated about *N* by a chronograph motor through a belt *Z*. The current for *P* was supplied through wires *S*, *S*, dipping into circular troughs of mercury in the ebonite block *T*. By this device the bearing for the plate *K* was nowhere exposed to atmospheric pressure and required no packing.

At the instant of releasing the spheres the wheel *Q* was set in steady rotation by putting it in gear with the chronograph motor, and as the spheres arrived successively at the bottom of the fall tube they were

received at different angular positions on the surface of the plate K. A chronograph pen connected in series with a seconds pendulum, rested on a paper strip R attached to Q, and marked each second as the wheel rotated. This time record is shown in plan just to the right of T, and also in drawing II for a different speed of rotation. After completing the experiment and stopping the rotation of Q, an index V, which fit over the edge of the case L was placed exactly 180° from the place the pen had ceased to mark. When this index pointed to any mark on the pen record as Q was rotated, the spheres which had fallen on K just as that mark was being made would be 180° from the bottom of the fall tube; that is they would be in a position to be seen through the window at F. By counting seconds from the beginning of the pen record to the index the elapsed time of fall was given directly.

This is a longer time than that which would be required if the sphere had started to fall with its terminal velocity, instead of starting from rest. The correction to be applied can easily be found. The following table gives its value for the apparatus used.

Observed Time. Seconds.	Correction. Seconds.
.50	−.06
1.00	−.03
2.00	−.02
3.00	−.01
6.00	−.01
7.00	0

These corrections have been applied to all times of fall less than seven seconds.

5. *Radius of Spheres.*—The radius a of each sphere was determined by measuring its diameter from two to eight times, the majority of the spheres being measured six times. An improved microscope plate micrometer[1] was used except in experiments II., 1, 2 and 3, and for fourteen spheres in other experiments which were too large to be measured by the plate micrometer used. These excepted spheres were measured by an ocular micrometer. The micrometer field was illuminated by diffused light reflected from the mirror Y (drawing IV, Fig. 3), through the window X and disc K. Test measurements using different magnifying powers gave concordant results for all the sizes used, and it appears that the measurements of radii, even for the smallest spheres, are not affected by constant errors due to diffraction, of an amount exceeding a small fraction of a wave-length of light. Some difficulty in the measurement of the smaller spheres was experienced on account of tremors of the

[1] John Zeleny and L. W. McKeehan, PHYS. REV., Vol. XXXII., p. 530.

pier on which the apparatus stood. The trouble was eliminated by measuring diameters after midnight, when heavy traffic on the streets had ceased. The method of making exterior contact of the cross-hair and the circular image of a sphere was found the most convenient for rapid measurement. A correction for the width of the cross-hair had therefore to be applied to all diameters measured. The amount of this correction was determined by using other forms of contact for comparison. with other objects than spheres.

6. *Pressure.*—The pressure inside the case L, which was of brass about 5 mm. thick, was reduced by a water pump or by a Gaede mercury pump, depending upon the reduction desired. The pumps were attached at U (Fig. 3). The window X was of plate glass set in paraffin which made a permanently air-tight joint. The cover of the case L was ground with emery to fit against the lower part, and a thin coating of vacuum wax between the surfaces gave an excellent joint. A platinum wire for the electrical connection to the solenoid in the fall box was sealed through the glass manometer tube outside the case, the return circuit being through the metal parts of the apparatus and its supports W, W.

Pressure readings were taken by means of a cathetometer on a closed-arm mercury manometer, and when the pressure was low enough it was also measured by a McLeod gauge with a small factor. Glass stopcocks served to isolate the apparatus from the pumps. The apparatus was left to itself for at least three hours just preceding the releasing of the spheres, so that the pressure and temperature of all its parts was steady. A dish containing P_2O_5 was placed inside the apparatus before reducing the pressure. The drying agent was considered necessary, although experiments at atmospheric pressure showed the difference in viscosity due to a small amount of water vapor was not appreciable. At low pressures the same amount of vapor, forming a larger fraction of the whole gas, might change the viscosity considerably, since its viscosity is much less than that of air.

7. *Method of Finding and Measuring Spheres.*—In finding and measuring the spheres the microscope was placed in guides which permitted it to move only along a radius of the plate K, 180° from that passing under the fall tube. The distance from the center of the plate was controlled by a slow motion screw. With the microscope focused on the surface of K, the wheel Q and plate K were very slowly turned by hand through suitable gearing, and the spheres were measured as found. Each sphere was placed near the center of the field by moving the plate or the microscope, and the current for the magnet P was interrupted during the measurements in order to prevent creeping of the plate K. After sur-

veying the whole of one strip or zone the microscope was moved radially the width of the field, and the process was repeated until enough spheres had been measured to show the variation of time of fall with radius, or until no more could be found. The radii for an experiment were not computed from the plate micrometer readings until all had been measured. The total number of spheres that fell to the plate was not greater than three hundred in any one experiment, so that no effect on the velocity due to mutual action between the spheres occurred.[1]

§ IV. Results of Experiments.

In the tables and curves *I* to *XVI*[2] are given the results of the experiments, arranged in order of decreasing pressure. The quantities tabulated and plotted are the radii in cm. $\times 10^{-4}$, and the reciprocals of the terminal velocities, that is the times in seconds required to fall one centimeter at this velocity. On the curves the radii are ordinates and the times abscissae. A small circle is drawn for each sphere measured.

The method of release shown in drawing *VIII*, Fig. 3, was found the most satisfactory of about seven methods, probably because the paper used closed the top of the fall tube throughout the experiment, and so no air currents set up by the movement of parts of the mechanism in the fall box could disturb the air in the tube. Large spheres did not, however, adhere to the paper and were not obtained by this method.

The results at atmospheric pressure using this method are given in Table and curve *I*, except that the spheres for which the radius exceeds .0013 cm. were obtained by the method shown in drawing *VII*, Fig. 3. The results at atmospheric pressure using other methods of release than the one shown in drawing *VIII* are contained in table and curve *II*. In neither of these tables are any spheres given which fell in the first twentieth of the period of a complete rotation of the plate *K*. The largest spheres are from experiments where the time of a rotation was 20 seconds, the smaller ones from experiments where the time was from 200 to 1200 seconds. The error in time measurements is thus kept small throughout the whole range.

The range of sizes used in all the experiments was from $a = .0025$ cm. to $a = .00012$ cm., and the range of pressures from atmospheric pressure (740 mm.) down to .32 mm. of mercury. The total time of fall for the various sizes and pressures varied from .74 sec. to 1,072 sec.

The value of the constant *B* in the formula was computed for each sphere in experiments VI., X., XIII. and XV., at pressures of 8.28, 2.11,

[1] Cunningham, loc. cit.

[2] Some of these are omitted here for the sake of brevity.

0.69 and 0.34 mm. of mercury respectively. The average values of B for the four experiments were .00724, .00724, .00826 and .00729, giving a mean value

$$B = .0075 \pm 2,$$

corresponding to

$$A = 1.00 \pm 3.$$

The close agreement of the first, second, and fourth values is accidental, as seen by considering the probable errors in the experiments. The probable errors in the time of fall are large for the largest spheres, and the probable errors in the radius are several per cent. for the smallest spheres. The probable error in the average value of B for an experiment also depends upon that in the pressure, since this is not reduced in taking the average. The errors in pressure for these four experiments are different, since the manometer was used for the first and second, the McLeod gauge for the third and fourth. If the four values are weighted with reference to these probable errors, the weighted mean is found to be a little higher than the unweighted mean given above. These pressures were chosen for computing B because the change in the time of fall from that found for atmospheric pressure is large, and yet the total time of fall is not too small for accurate measurement, as it would be at still lower pressures.

Values of the time per centimeter were computed using the value of A corresponding to the assumption of normal emergence of the impinging molecules, $A = 1.05$, for each pressure used, and for the following values of the radius:

.004	.0005
.0035	.0004
.003	.0003
.0025	.00025
.002	.0002
.0017	.00018
.0014	.00015
.0012	.00014
.001	.00013
.0008	.00012
.0006	

These times are plotted on curves *I*, and *III* to *XVI*, Figs. 5 to 9, as small crosses, and it will be seen that the circles representing observed spheres agree well with a line passing through these crosses.

An idea of the great amount of variation in time of fall will be obtained by referring to curve *I*, Fig. 5. The black dot near the left of the figure represents the time of falling one centimeter for a sphere of radius .00014

cm., at a pressure of 0.32 mm. of mercury. At atmospheric pressure this sphere would fall in a time shown by the last cross at the right of the figure. The value of the term B/ap in the formula varies from .003 for a = .004 cm., p = 740 mm., to 196 for a = .00012 cm., p = .0.32 mm. The agreement of the experiments with the formula throughout the entire range is excellent.

The differences in computed velocities produced by using one or another value of A can be seen by referring to Fig. 4. Three curves are

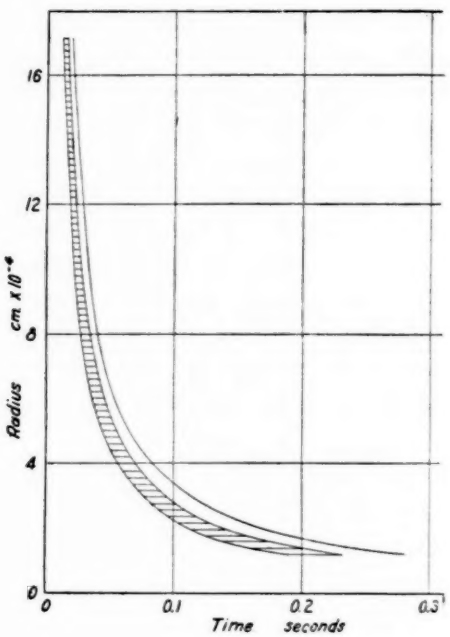

Fig. 4.

drawn, showing the variation of the time required to fall one centimeter (abscissæ) with the variation of the radius in cm. $\times$ 10^{-4} (ordinates), at a pressure of 0.32 mm. of mercury, and at a temperature of 20° C. The curve at the left, and the one connected to it by horizontal lines are drawn for A = 1.5 and A = 1.2, respectively, and correspond to the limiting values possible under Cunningham's assumptions. The curve at the right is drawn for A = 1.05, the value corresponding to the assumption of normal emergence of the impinging molecules. Curve *XVI*, Fig. 9, which shows the experimental results at this pressure, agrees almost exactly with this last curve.

At atmospheric pressure the differences are not detectable by the method used, for the range of sizes obtained. This will be seen by com-

paring curves *I* and *II*, Figs. 5 and 6. The crosses in *II* represent times computed by assuming $A = 0$ (Stokes's formula), those in *I* times computed by assuming $A = 1.05$. The difference amounts to 7 per cent. in time for the smallest sphere shown, being less for all larger spheres, and the additional difference in case $A = 1.5$ would be about 3 per cent., again for the smallest sphere shown. An error of 1.5 per cent. in the measurement of the radius of such a sphere would entirely mask this additional difference in time of fall.

The only other experimental value of the constant in Cunningham's

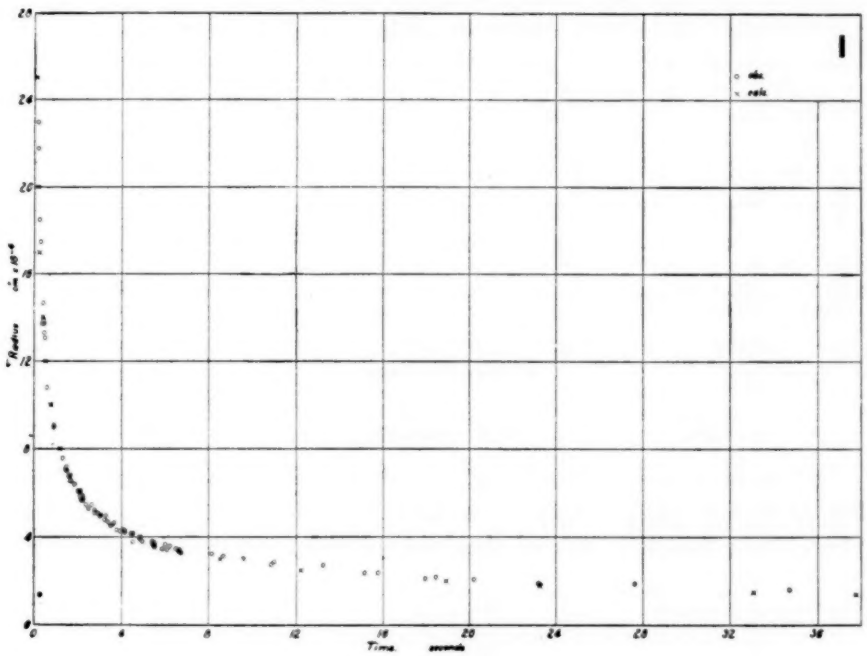

Fig. 5.

Atmospheric pressure.

formula is given by Professor R. A. Millikan,[1] who found A to be 0.815. This value is determined from experiments on spheres of different sizes, but all at one pressure, namely atmospheric. The method is indirect in that the radii of the spheres were not measured but were computed from their time of fall on the assumption of uniform electrical charge for all the spheres. The material used was a liquid, and internal eddies in the drops may cause a consumption of energy, and consequently reduce the apparent value of A.

[1] Science, Vol. XXXII., p. 446.

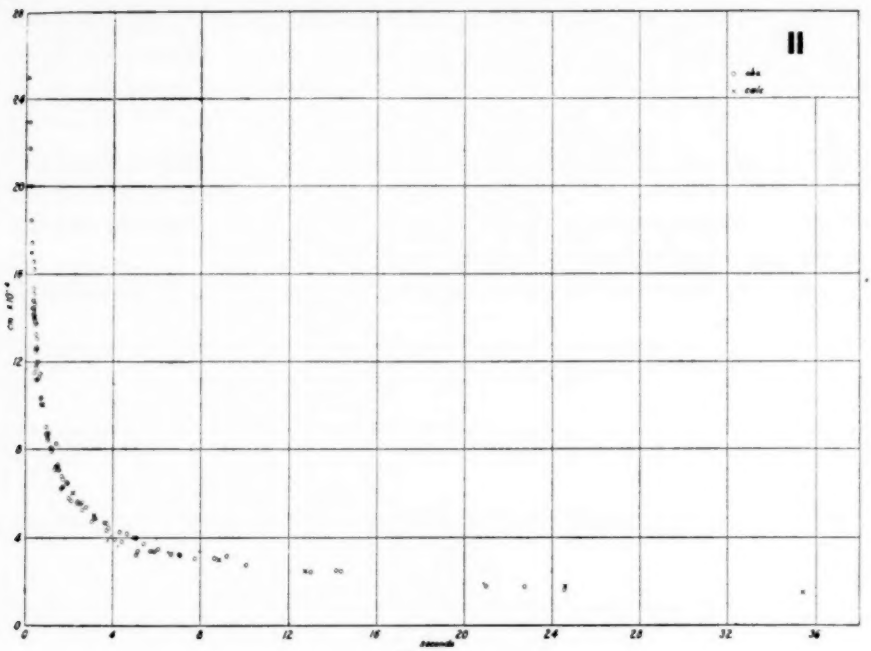

Fig. 6.

Atmospheric pressure.

§ V. SPORES.

Lycopodium spores, which fell at atmospheric pressure only about half as fast as Stokes's formula required, were found at lower pressures to maintain the same ratio of observed velocities to velocities computed

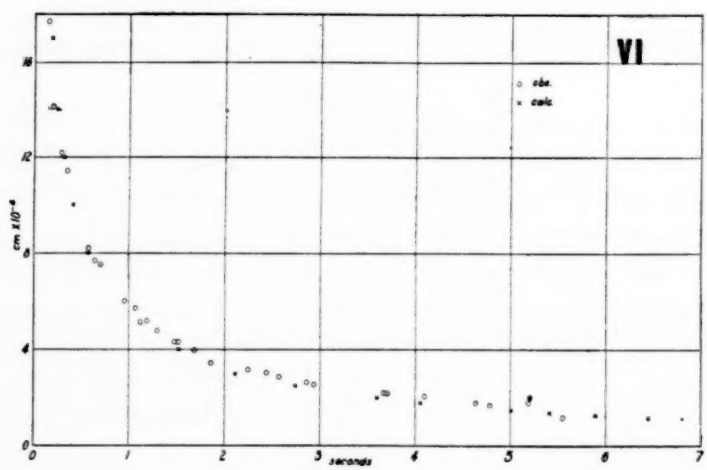

Fig. 7.

Pressure = 8.28 mm.

by the formula which held at all pressures for wax spheres. This means that the density, which is the only measured quantity appearing in the formula in the first degree, was very incorrect. The density determined by refined volumenometer measurements was 1.175. This may have been the density of a solid shell surrounding an air space or the density of a spongy mass containing many air spaces. The average density, meaning by this the quotient of the mass of a spore by the volume of a sphere having its mean radius, would have to be about 0.6 to make the spore behave as it does. In any event, turbulent motion of the air can no longer be held responsible for the variations observed at atmospheric pressure. Experiments with the other spores used before were not attempted, because the condition of the material appeared to have changed with time.

§ VI. Summary and Conclusion.

The formula

$$V = \frac{2}{9}\frac{ga^2(\sigma - \rho)}{\mu}\left(1 + A\frac{l}{a}\right),$$

in which A is a constant, expresses the terminal velocity of fall of small solid spheres in air throughout a wide range of radius and pressure. The

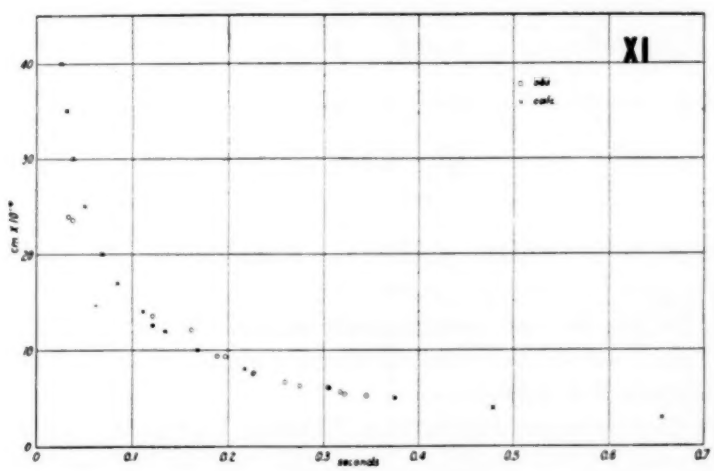

Fig. 8.

Pressure = 2.01 mm.

constant A appearing in this formula is found to have a theoretical value depending on the assumption consistent with the kinetic theory of gases which is made concerning the mode of impact of the gas molecules on the surface of the sphere. Assuming elastic impacts the value of A is 1.5;

assuming inelastic impacts and the same distribution of component velocities in the emerging as in the impinging molecules, the value of A is 1.2; assuming inelastic impacts and normal emergence of all impinging molecules, the value of A is 1.05; by experiment the value of A is 1.00 $\pm$ 3.

The close agreement of the experiments with the fomula derived for the assumption of normal emergence makes it probable that a gas molecule impinging on a solid is entangled in the surface layer of molecules, and emerges again after a number of collisions with these molecules, its direction of emergence being generally nearly normal to the surface. The

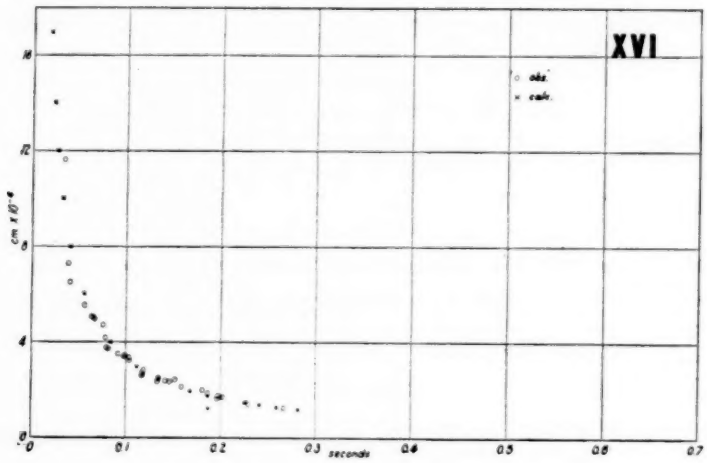

Fig. 9.

Pressure = 0.32 mm.

experiments do not indicate whether this penetrable layer is composed of the same kind of molecules as the rest of the solid, or is a condensed layer of the gas, but only show that the distances between the molecules in it are small compared to the distances in a gas under normal conditions of temperature and pressure.

I take great pleasure in thanking Professor John Zeleny for his continuous interest in the progress of the work, and for his many valuable suggestions on the solution of experimental and theoretical difficulties.

Physical Laboratory,
University of Minnesota,
May 17, 1911.

TABLE I.

Experiment 1. *p* 736 *mm. Temperature* 20°.6 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 20 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.002293	.1929	.001746	.297	.001369	.481
.002170	.1995	.001461	.419	.001326	.474
.001846	.2404	.001382	.437	.001308	.518

Experiment 2. *p* 749 *mm. Temperature* 18°.3 *C. Tube length* 30.96 *cm. Maximum time of fall measurable* 226 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.001369	.379	.000560	2.296	.000416	4.56
.001077	.600	.000541	2.45	.000407	4.51
.000906	.934	.000541	2.70	.000393	4 86
.000898	.934	.000522	2.56	.000385	4.97
.000757	1.365	.000505	2.86	.000385	5.40
.000719	1.524	.000492	3.36	.000382	4.52
.000696	1.566	.000466	3.47	.000373	5.45
.000679	1.718	.000461	3.75	.000368	5.49
.000674	1.653	.000453	3.65	.000355	5.52
.000653	1.748	.000449	3.59	.000350	5.88
.000637	1.916	.000433	4.10	.000346	6.10
.000608	2.169	.000432	3.86	.000346	6.67
.000604	2.088	.000427	4.18	.000339	6.63
.000585	2.270	.000426	3.96	.000335	6.72
.000580	2.189	.000417	4.18	.000334	6.79

Experiment 3. *p* 740 *mm. Temperature* 19°.7 *C. Tube length* 30.96 *cm. Maximum time of fall measurable* 1,080 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.000704	1.492	.000370	6.03	.000238	15.18
.000571	2.335	.000361	6.26	.000238	15.79
.000562	2.205	.000351	6.52	.000218	18.49
.000518	2.854	.000327	8.17	.000213	17.96
.000502	2.984	.000317	8.69	.000210	20.24
.000490	3.081	.000303	9.63	.000192	23.22
.000473	3.31	.000287	11.03	.000191	27.64
.000413	4.44	.000274	10.86	.000162	34.76
.000378	5.52	.000273	13.26		

Table III.

Experiment 1. p 100.4 *mm. Temperature* 21°.0 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 212 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.002660	.1462	.001176	.520	.000529	2.52
.002177	.2046	.001068	.695	.000497	2.95
.002069	.2795	.000992	.754	.000497	3.25
.001918	.296	.000833	1.170	.000440	3.49
.001854	.221	.000793	1.011	.000438	3.87
.001840	.260	.000712	1.436	.000320	6.76
.001531	.318	.000641	1.781	.000313	6.60
.001293	.482	.000543	2.64	.000302	6.88
.001207	.497	.000537	2.38		

Table IV.

Experiment 2. p 34.85 *mm. Temperature* 21°.2 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 186 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.003715	.0715	.001013	.676	.000634	1.501
.002989	.1040	.000975	.669	.000623	1.689
.002642	.1040	.000926	.767	.000532	2.053
.002591	.1234	.000909	.786	.000516	2.339
.001506	.351	.000908	.812	.000487	2.385
.001347	.429	.000808	1.027	.000460	2.781
.001199	.487	.000807	1.053	.000380	4.14
.001109	.604	.000737	1.170	.000331	5.10
.001095	.617	.000635	1.527	.000330	4.71

Table VI.

p 8.28 *mm. Temperature* 21°.3 *C. Tube length* 30.96 *cm. Maximum time of fall measurable* 172 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.001769	.1427	.000517	1.118	.000263	2.86
.001411	.1978	.000477	1.287	.000257	2.94
.001216	.292	.000430	1.472	.000220	3.67
.001141	.353	.000429	1.517	.000219	3.71
.000819	.584	.000399	1.686	.000204	4.10
.000766	.652	.000343	1.861	.000175	4.64
.000752	.712	.000316	2.250	.000175	5.18
.000599	.960	.000303	2.445	.000169	4.79
.000569	1.067	.000287	2.57	.000119	5.54
.000518	1.183				

TABLE VII.

Experiment 2. *p* 7.70 *mm. Temperature* 20°.9 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 184 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.002090	.1787	.000764	.650	.000466	1.495
.002038	.1787	.000704	.689	.000382	1.589
.001375	.276	.000701	.712	.000327	1.949
.001126	.370	.000688	.731	.000316	1.888
.001065	.491	.000621	.896	.000307	2.066
.001059	.445	.000555	.965	.000306	2.129
.001044	.406	.000525	1.104	.000274	2.556
.000922	.478	.000514	1.072	.000257	2.550
.000916	.500	.000510	1.153	.000210	3.17
.000873	.533	.000507	1.004	.000179	3.12
.000784	.630	.000494	1.134		

TABLE IX.

Experiment 1. *p* 3.14 *mm. Temperature* 20°.2 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 20 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.002042	.0809	.001268	.1832	.000699	.390
.002027	.0923	.001240	.1914	.000687	.419
.002007	.0939	.001145	.1654	.000658	.401
.001966	.1134	.001106	.2012	.000609	.418
.001790	.1150	.001038	.2109	.000565	.460
.001775	.1118	.000930	.2275	.000538	.487
.001372	.1556	.000892	.2616	.000502	.496
.001365	.1540	.000890	.289	.000456	.565
.001327	.1703	.000709	.341	.000321	.625

TABLE XI.

Experiment 1. *p* 2.01 *mm. Temperature* 20°.6 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 20 *sec.*

a	$1/V$
.002396	.0341
.002358	.0387

Experiment 2. *p* 2.01 *mm. Temperature* 20°.5 *C. Tube length* 30.9 *cm. Maximum time of fall measurable* 19 *sec.*

a	$1/V$	a	$1/V$	a	$1/V$
.001357	.1208	.000768	.2278	.000624	.304
.001213	.1622	.000764	.2274	.000569	.317
.000940	.1888	.000670	.2596	.000515	.322
.000931	.1969	.000626	.275	.000510	.345

TABLE XII.

p 1.28 *mm.* *Temperature* 20°.7 *C.* *Tube length* 30.9 *cm.* *Maximum time of fall measurable* 18 *sec.*

a	1/*V*	*a*	1/*V*	*a*	1/*V*
.002222	.0426	.000808	.1270	.000391	.287
.001616	.0647	.000752	.1494	.000364	.311
.001533	.0672	.000623	.1595	.000340	.348
.001422	.0738	.000588	.1914	.000304	.340
.001323	.0838	.000554	.2034	.000294	.399
.001320	.0844	.000512	.259	.000293	.420
.001164	.0893	.000408	.280	.000281	.392
.001049	.1082	.000401	.324	.000245	.457
.001033	.0910	.000394	.321	.000233	.548
.000883	.1248				

TABLE XVI.

p 0.32 *mm.* *Temperature* 21°.3 *C.* *Tube length* 30.96 *cm.* *Maximum time of fall measurable* 17 *sec.*

a	1/*V*	*a*	1/*V*	*a*	1/*V*
.001163	.0341	.000349	.0992	.000239	.1472
.000730	.0379	.000347	.0973	.000216	.1598
.000647	.0405	.000340	.1028	.000202	.1816
.000548	.0564	.000327	.1031	.000190	.1871
.000501	.0638	.000289	.1177	.000178	.1968
.000494	.0668	.000273	.1167	.000172	.2024
.000468	.0752	.000263	.1161	.000163	.1961
.000415	.0788	.000253	.1346	.000142	.228
.000381	.0798	.000244	.1524	.000127	.266
.000379	.0814	.000242	.1417		
.000357	.0914	.000239	.1333		

HEAT OF EVAPORATION OF WATER.

By Arthur Whitmore Smith.

THE "heat of evaporation" of water means the heat per gram required to change water into saturated vapor at the same temperature. Various determinations of this value have been made during the past century,[1] the most extensive investigation being the work of Regnault[2] over sixty years ago. In the older methods steam from a boiler was passed directly into a calorimeter, where it was condensed. The heat of condensation was measured by the increase in temperature of the cold water of the calorimeter, while the amount of vapor was determined by weighing the condensed water. In more recent experiments the water was boiled in the calorimeter, a known amount of heat being supplied by an electric current. The amount of water thus removed from the calorimeter was determined either by the loss in weight of the containing vessel,[3] or by condensing the steam and weighing the water thus collected.[4]

It is now well known among mechanical engineers that the vapor rising from boiling water carries with it a certain amount of water in a finely divided state, but nevertheless still liquid. The amount of water thus carried in the steam is often over one per cent., and it may be several per cent. of the whole. This leads one to inquire what quality of steam was used in previous experiments to determine the heat of evaporation. If the steam used contained 1 per cent. of moisture it is evident that the familiar number 537 is 1 per cent. too low.

Regnault appears to have known something of the difficulty of obtaining dry steam, and in his classical experiments collected the steam by means of a long tube coiled in the upper part of his boiler. It is now known that this precaution is not sufficient. The later determinations by Joly[5] with a "steam calorimeter," and by Henning vigorously boiling water in a small calorimeter[6] with no separator, certainly were made with steam containing an uncertain amount of moisture.

[1] See Preston, Theory of Heat, p. 378 ff.

[2] Regnault, Institute de France, Mem. Acad. Sci., Vol. 21, pp. 1–748, 1847.

[3] Griffiths, Phil. Trans., Vol. 186 A, pp. 261–342, 1895.

[4] A. W. Smith, Phys. Rev., Vol. 25, pp. 147–170, 1907. Henning, Ann. der Phys., Vol. 21, pp. 849–878, 1906; Vol. 29, pp. 441–465, 1909.

[5] Joly, Phil. Trans., Vol. 186 A, note on p. 322, 1895.

[6] Henning, loc. cit., Fig. 1, p. 446.

It is the object of this research to determine the heat of evaporation of water at the boiling point more exactly than has yet been done, and especially to eliminate this serious error which apparently has gone unnoticed. The practical need of knowing the exact value of this constant is evident, since it is the basis of the steam tables used in all problems in steam engineering, and a change in this value means a corresponding change throughout the steam tables.

The relative values of the heat of evaporation at different temperatures have been well established by the recent investigation by Davis[1] on the results obtained in experiments on the free expansion of dry steam. Unfortunately the absolute value at any point on his curve is unknown, and steam tables[2] based on this curve assume for the value at 100° C. the mean of the results obtained by Joly and by Henning.

If a reliable determination of the heat of evaporation can be obtained at any one temperature it will have the effect, through this curve, of establishing its value throughout the range of temperatures used in steam engineering.

As far as I am aware, no attempt has ever been made to determine the heat of evaporation from a quiet surface of water. Evaporation from such a smooth surface is supposed to take place by one molecule at a time leaving the liquid and going into the vapor. The action inside of a steam boiler is very different from this, and it would be surprising not to find some fine spray thrown into the steam. It is from this point of view that the problem has now been attacked.

METHOD.

A gentle stream of dry air is drawn through the calorimeter, passing over the unruffled surface of the hot water and carrying away some of the vapor. Near the boiling point water evaporates very readily, and care must be exercised not to overcool the calorimeter by too rapid evaporation. Heat is supplied by an electric current, and the air current is constantly regulated to allow the evaporation to proceed just fast enough to keep the temperature constant at 98° C. After passing out of the calorimeter most of the vapor is condensed in a cold flask, while the remainder is caught by drawing the air through sulphuric acid. The gain in weight gives the amount of water evaporated in the calorimeter.

The heat received by the calorimeter is that generated in the heating coil plus a small amount which comes from the surroundings by radiation

[1] Davis, Journ. Am. Soc. Mech. Eng., No. 1212, Vol. —, pp. 1419–1452.

[2] Marks and Davis, Steam Tables, Longmans, 1909.

and conduction. The latter cannot be measured directly, but it can be eliminated in the computations by the following method. Each result is computed from two experiments, in one of which a large amount, and in the other a much smaller amount of water is evaporated. The temperatures and other conditions are the same in each and so also is the amount of heat received from the outside. Therefore the difference between the two total amounts of heat eliminates these corrections and is the same as the difference between the two amounts of heat delivered by the current alone. This difference, divided by the difference between the corresponding amounts of water, gives the heat per gram necessary to evaporate water at this temperature.

Thus let M' denote the number of grams of steam produced in a given experiment, H' the corresponding amount of heat supplied by the current, and h the small amount of heat gained by radiation, conduction, etc. Then

$$H' + h = LM', \tag{1}$$

where L is the heat of evaporation. On another day the experiments are repeated under the same conditions as before, but using a larger amount of heat. Then

$$H'' + h = LM'', \tag{2}$$

and combining these two equations by subtraction gives

$$L = \frac{H'' - H'}{M'' - M'}. \tag{3}$$

All corrections for heat gained by radiation or conduction, whether known or unknown, are eliminated by this method of differences. There were no starting or stopping corrections, as each experiment, so called, was a one hour section from the middle of an all day run. The calorimeter, C, and all of its surroundings, shown in Fig. 1, were kept hot, day and night, by the steam-bath. Early in the morning the calorimeter was filled with hot distilled water and maintained at 98° C. all of the forenoon. After noon the heating current was started and the air regulated to hold the temperature constant at the same point. Only after a preliminary run of an hour or so, and when everything was running smoothly and constant, was the steam directed into one of the weighed flasks, and an experiment begun. A second and third experiment might follow, but not more than three runs were made during one day.

Variations.—All of the experiments were made at 98° C., as it was necessary to keep below the boiling point. Since the latter varied from

day to day over a range of nearly one degree, the corrections due to heat received by radiation, etc., varied by 100 per cent. Sometimes a series of experiments in each of which a large amount of water was evaporated, were made on one day; and a similar series of check experiments in each of which a small amount of water was evaporated were made on another day when all other conditions were as nearly the same as possible. At other times such check experiments were made at the beginning of the afternoon, and immediately followed by the experiments with larger heats. Again, the check experiments followed the regular experiments. Thus in all respects were the check experiments performed under the same conditions as the others, except in the amounts of water evaporated.

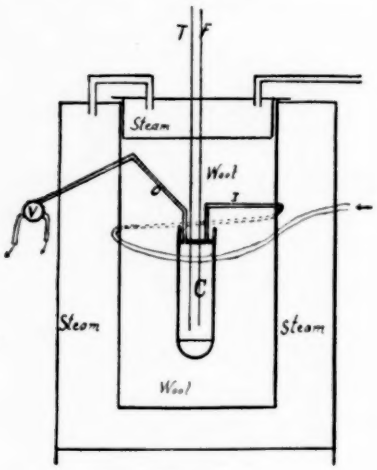

Fig. 1.

Calorimeter and steam jacket.

THE CALORIMETER.

The calorimeter consists of a thermos bottle, or vacuum walled jar, fitted with a copper lining which serves the double purpose of equalizing the temperature between the top and bottom and enabling the connecting tubes to be soldered in air tight. The internal arrangement is shown in Fig. 2.

The heating coil is made of about one ohm of manganin wire wound on a brass tube, and well insulated with silk and shellac, the whole being enclosed within a larger tube so that no water could come in direct contact with any part of the coil. The leads, LL', were of heavy copper wire to reduce the amount of heat produced in them by the current, and which might then be conducted into the calorimeter. Potential leads, pp', were soldered to the main leads close to the top of the calorimeter.

The glass tube, F, for filling and emptying the calorimeter, and the thermometer, T, were passed through snug fitting brass tubes which extended well into the water, thus completely sealing the joint. The rotary stirrer drew the water up through the tube on which the heating coil was wound. Friction was very small as it was mounted on glass bearings, and conduction of heat along the stirrer rod, S, was reduced to a minimum by making it of wood. The brass tubes, I and O, for

conveying the air current in and out were long and very thin-walled in order that the heat conduction might be as small as possible.

The incoming air was directed downward upon the surface of the water, while the outgoing stream of air and vapor passed through a coil in the water to warm it, if necessary, to the same temperature. Whatever small difference of temperature there may be between the incoming and outgoing air is measured by thermal junctions, *Th*, inserted into the tubes near the top of the calorimeter and corrections made therefor.

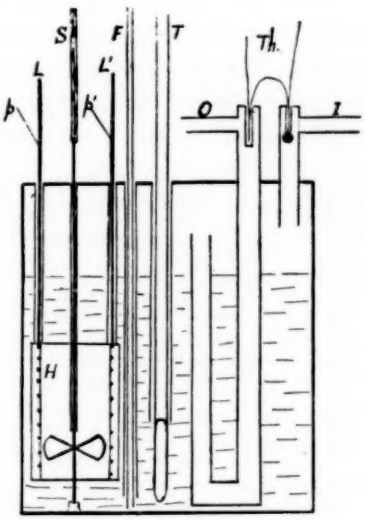

Fig. 2.

Interior of the calorimeter.

The Thermometer.

The temperature of the water in the calorimeter was measured by the thermometer, *T*, which was graduated on the stem to hundredths of a degree, and could be read to thousandths. It was kept in the calorimeter all of the time and thus not subjected to great changes of temperature during the course of these experiments.

The readings of this thermometer do not enter into any computation, and the final result is independent of all temperature measurement. The thermometer serves merely as an indicator to enable the observer to keep the calorimeter at a constant temperature. The only use made of the actual reading is in determining the temperature at which the water was evaporated. The 98 degree mark was chosen as the constant temperature at which to hold the calorimeter. The upper fixed point of the thermometer was carefully determined by means of a regular hypsometer, the test being made in the same room and with the same stem correction as when the thermometer was used. The corrected value for the 98 degree mark was thus determined to be 98.07° C.

Electric Heating.

The current for the heating coil was supplied from three sets of storage cells in parallel, and after the first two hours was remarkably steady. often maintaining five amperes without a change in the potentiometer for an hour. Both the current and the fall of potential over the heating

coil were measured in terms of a Weston normal cell, and the heat computed from the formula,

$$\text{Heat} = EITJ.$$

The electrical connections are shown in diagram by Fig. 3. The heating coil, H, is in series with a standard one-fourth ohm coil, C, the storage battery, a variable resistance, v, and an ammeter. The current was set at about the desired value and allowed to flow undisturbed. In parallel with H are two accurate high resistance boxes, P and Q. P remains set at 10,000 ohms, while Q is varied so as to maintain the fall of potential over P equal to the E.M.F. of the standard cell, E_s. Then the fall of potential over H, being the same as over P and Q together, is

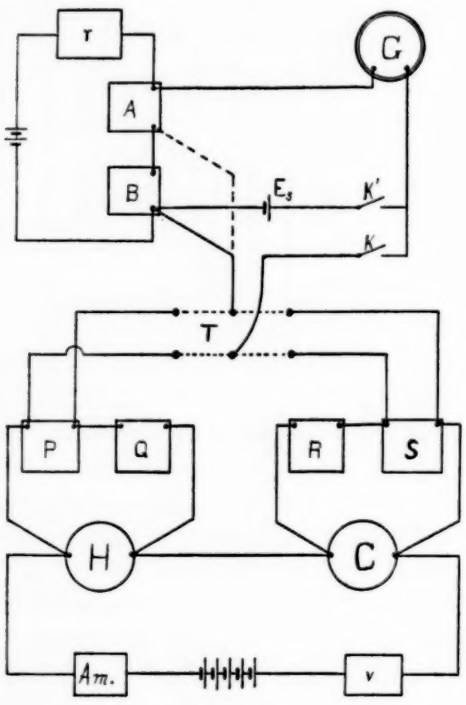

Fig. 3.

Electrical connections.

$$E = \frac{P+Q}{P} E_s.$$

As the constant use of the standard cell would be liable to polarize it, this comparison is made by substitution, the fall of potential over AB in the auxiliary circuit ABr, being first balanced against th standard cell, using key K', and varying r; then this fall of potential is balanced against that over P, using key K and adjusting Q.

In the check experiments, where the heating current is only half as large, the cell is balanced against both A and B, which are 10,000 ohms each, and then the fall of potential over P is balanced against that over A alone, as shown by the dotted connection.

The current is determined by measuring the fall of potential E', over the standard resistance in the same way, by means of the double throw switch T, and its value is computed by the relation,

$$I = \frac{E'}{C} = \frac{R+S}{CS} E_s.$$

The total current is larger than this by the shunt current through R and S; and for the same reason the current through the heating coil is smaller than the total current by the current through the shunt P and Q. These shunt currents are made equal by keeping P and S each set at 10,000 ohms, and thus the current through H is strictly equal to that through C. The resistances enter as ratios of coils in the same box and therefore require no temperature correction. The ratios were calibrated by comparison with standard coils from the Reichsanstalt. The exact values of E, and C were determined by the Bureau of Standards.

Measurement of Time.—Each run was for a period of one hour, this interval being measured by a standard Riefler clock. The connection between clock and calorimeter was made by means of a high grade stop watch with a split seconds hand. Just on the hour by the clock, the time was recorded on the watch by stopping the lower half of the moving seconds hand. One minute later, with the watch held in the left hand and the right hand on the distributing valve, V, of the calorimeter, both hands were turned, one stopping the upper half of the seconds hand at the same time that the other turned the valve. Turning this valve ended one run and began the next. The exact time at which this was done could then be read from the watch to the nearest fifth of a second, and this was usually an even minute after the reading of the clock.

The Mechanical Equivalent of Heat.—The energy delivered to the calorimeter by the current is EIT joules. When it comes to translating this into heat units (calories) it is necessary to use the factor known as the mechanical equivalent of heat. And since the specific heat of water is not constant, but has a different value at each temperature, it is necessary to define precisely what is meant by the term "calorie." The unit used in this paper is the "mean calorie," that is, one per cent. of the heat that is required to warm one gram of pure water from 0° C. to 100° C.

The best direct determination of the mechanical equivalent of heat is doubtless that of Reynolds and Moorby[1] in which they found

$$1 \text{ mean calorie} = 4.1836 \times 10^7 \text{ ergs.}^2$$

Probably the best determination of this constant by the electrical method is that of Callendar and Barnes[3] who found

$$1 \text{ mean calorie} = 4.1849 \text{ joules,}^4$$

[1] Reynolds and Moorby, Phil. Trans., Vol. 190 A, pp. 301–422, 1897.

[2] For a more complete discussion of this subject see paper by A. W. Smith, United States Weather Review, Vol. 35, pp. 458–463.

[3] Callendar and Barnes, Phil. Trans., 1902.

[4] Barnes, Proc. Roy. Soc. Lond., Vol. 82, p. 390, 1909.

provided the E.M.F. of their Clark cells be taken as 1.4333 volts at 15° C. This is equivalent[4] to taking the E.M.F. of the standard Clark cell as 1.4330 at 15° C. or the Weston normal cell as 1.0187 volts at 20° C.

Since January 1, 1911, the E.M.F. of the Weston normal cell is taken as 1.0183 volts at 20° C., and all of my measurements are made in terms of this unit. Therefore, since the electrical determination of J includes E^2, and any reduction in the number expressing the latter will reduce the former twice as much, the value for the mechanical equivalent of heat which should be used in connection with the value of 1.0183 for the Weston normal cell is

$$4.1849 \times \left(\frac{1.0183}{1.0187}\right)^2 = 4.1816.$$

The mean between these two values is 4.1826, and this is the value used in reducing my measurements to mean calories.

COLLECTING THE WATER.

The most difficult part of the work was the determination of the amount of water evaporated. While the value of the electric current can be accurately measured at any instant, the only way to find the corresponding rate of steam production is to collect the steam for an hour or so and weigh it. During this period, and for several hours preceding it, the rate of evaporation and the temperature of the calorimeter must be maintained constant, and this means continuous observation of the thermometer and adjustment of the air current. This is tedious, but necessary.

After leaving the calorimeter the stream of air and vapor passes through a two way valve, V, Fig. 1, by which it can be directed into either one of two sets of collecting tubes. In order that there should be no condensation of vapor in this valve or the connecting tubes they were enclosed in a box with an electric lamp which maintained the temperature at 130° C. The steam then passed into a half liter glass flask in a cold water-bath, where most of it was condensed. The remaining vapor, with the air, passed through three tubes containing H_2SO_4 where the water was all caught. (The incoming air was likewise dried by passing through H_2SO_4.) The acid, of density 1.84, was very efficient in collecting the water, practically none of it getting through the first tube. As soon as the second tube began to catch water the first tube was refilled with fresh acid.

Weighing the Tubes.—The tubes and bulbs in which the water was caught were allowed to stand some time to assume room temperature.

They were then carefully wiped with a clean towel and weighed on a sensitive balance. All weighings were made with the aid of counterpoise consisting of a glass tube or bulb as nearly as possible like the one being weighed and likewise containing acid or water, as the case might be. All the tubes were kept closed except when on the balance; then the closing plug was a glass tube nearly sealed off. This allowed the pressure inside the tube to become the same as that outside, which is very important. The weighings were corrected for the buoyancy of the air, and to the weight of water thus determined was added the weight of the vapor filling the space in the calorimeter that had been occupied by the evaporated water.

RESULTS.

The results of these experiments are shown in Table I. Each value is computed from the results of two separate experiments. In the first column is shown the amount of water evaporated during one hour by a

TABLE I.

Heat of Evaporation of Water from a Smooth Surface at 98.07° C.

Water Evaporated (Grams).			Heat Supplied (Calories).			Calories Per Gram.
Large.	Small.	Difference.	Large.	Small.	Difference.	
37.3531	9.8858	27.4673	20,354	5,340	14,914	542.98
37.7645	9.1421	28.6224	20,379	4,888	15,491	541.23
37.7314	9.1132	28.6182	20,347	4,875	15,472	540.64
37.7146	9.1421	28.5725	20,375	4,888	15,487	542.03
37.5230	9.8858	27.6372	20,328	5,340	14,988	542.32
37.6916	9.8805	27.8111	20,379	5,320	15,059	541.48
37.6609	9.8805	27.7804	20,387	5,320	15,067	542.37
36.7130	9.3242	27.3888	19,816	4,974	14,842	542.15
36.6852	9.3242	27.3610	19,786	4,974	14,812	541.36
36.6826	9.3242	27.3584	19,784	4,974	14,810	541.34
36.5830	9.1421	27.4409	19,786	4,888	14,898	542.91
36.4116	9.1132	27.2984	19,689	4,875	14,814	542.68
36.4976	9.1421	27.3555	19,744	4,888	14,856	543.08
40.2116	8.6613	31.5304	21,686	4,582	17,104	542.12
36.8091	9.3242	27.4849	19,866	4,974	14,892	541.82
36.7000	8.6612	28.0388	19,750	4,582	15,168	540.96
37.3713	9.3242	28.0471	20,166	4,974	15,192	541.66
36.7925	8.9111	27.8814	19,878	4,774	15,104	541.73
36.1450	8.9735	27.1715	19,676	4,955	14,721	541.79
36.0249	8.9735	27.0514	19,612	4,955	14,657	541.83
					Mean	541.93

current of about five amperes. In the second column is given the amount of water evaporated by about half as large a current during an hour on

another day when all the other conditions, as nearly as possible, were the same as on the first day. The difference between these amounts is given in the third column. The corresponding amounts of heat supplied to the calorimeter and their difference, are shown in the three succeeding columns.

The values in the last column are computed from these differences by formula (3), and each one is an independent determination of the heat of evaporation of water at 98.07° C. The mean of these values gives 541.93 mean calories per gram. At 100° C. this would be 1.23 less, or

$$L_{100} = 540.70 \text{ mean calories per gram.}$$

The probable error of this result, computed by the method of least squares from the variations among the individual determinations, is 0.025 per cent.

The work of Davis referred to above leads to the relation,

$$H_t = H_{100} + .3745\ (t - 100) - .000990\ (t - 100)^2,$$

where H_{100} denotes the unknown total heat (above 0° C.) of evaporation at 100° C., and H_t is its value at t° C. We can now write

$$H_t = 640.7 + .3745\ (t - 100) - .000990\ (t - 100)^2,$$

which gives the value of the total heat of evaporation, expressed in mean calories, at any desired temperature t° C.

TABLE II.

Heat of Evaporation of Water when Boiling at 100° C.

Water Evaporated (Grams).			Heat Supplied (Cals).			Calories per Gram.	Temp.	Heat of Evaporation at 100° C.
Large.	Small.	Diff.	Large.	Small.	Diff.			
36.6929	8.9916	27.7013	19,894	4,979	14,915	538.4	99.30	538.0
36.7879	8.9693	27.8186	19,905	4,946	14,959	537.7	99.30	537.3
36.0010	8.4318	27.5692	19,440	4,680	14,760	535.4	99.64	535.2
35.7199	8.4318	27.2881	19,316	4,680	14,636	536.4	99.66	536.2
							Mean	536.7

While this result is about one per cent. larger than the old value, it is not as much larger as might have been expected if we are to believe the current statements that steam often carries several per cent. of moisture. Nor is there much chance of any constant error due to this method, or the apparatus, as a number of experiments were made in precisely the same way as those described above except that the air

current was much reduced. This allowed the water to boil, while the air current was only enough to keep the steam moving in the right direction and to prevent its working back and condensing in the incoming air tube. The data from these experiments are shown in Table II., and the final result, reduced to 100° C., gives 536.7 mean calories per gram. This shows that the old value can be obtained with this apparatus by using the old method and allowing the water to boil. By merely checking the boiling and causing the evaporation to take place from the smooth surface of the water the heat of evaporation is increased by 4 calories per gram. This is probably due to the fact that the steam leaving the calorimeter in the latter case is all vapor, while in the former case some liquid is also carried away and counted as vapor.

PHYSICAL LABORATORY,
UNIVERSITY OF MICHIGAN,
May 11, 1911.

THE DISTRIBUTION OF CURRENT AND THE VARIATION OF RESISTANCE IN LINEAR CONDUCTORS OF SQUARE AND RECTANGULAR CROSS-SECTION WHEN CARRYING ALTERNATING CURRENTS OF HIGH FREQUENCY.

BY HIRAM WHEELER EDWARDS.

I. INTRODUCTION.

THE present paper undertakes the investigation of the virtual resistance, inductance and current distribution of long, straight conductors of square and rectangular cross-section, when carrying alternating currents of high frequency. Experimental observations are given, which, within certain limits, corroborate the calculations for virtual resistance, but no attempt was made to measure the current intensity and inductance. Approximate formulas are derived by means of which the virtual resistance, the inductance and the current intensity may be calculated for particular cases.

The problem of determining the virtual resistance and inductance of long, straight, isolated conductors, when carrying alternating current is not a new one. In the oscillating current circuit of wireless telegraph, the increase of resistance due to the high frequencies of alternation is a very impotrant factor. The Tesla experiments illustrating the "skin-effect" show that in a number of cases the virtual resistance is many times the resistance offered to direct current. In the alternating-current railroads, the use of the rails for the return circuit is impracticable because of the increased resistance.[1]

For long, straight, cylindrical wires, Maxwell[2] has developed an expression for the virtual resistance and inductance. Rayleigh[3] has modified Maxwell's results by showing that the permeability of the material of the conductor, if greater than unity, causes a further increase in the resistance. Kelvin[4] solved the problem in a different manner, obtaining results similar, in final form, to those of Maxwell. He has given simplified forms of expression for virtual resistance and inductance for the

[1] Standard Handbook, for El. Eng., Sec. 2, No. 122; Sec. 11, No. 33.

[2] Maxwell, E. & M., Vol. 2, third ed., p. 320. See also Gray, E. &. M., Vol. 2, Pt. 1, p. 325; and Thomson, Recent Researches in E. &. M., Chap. 4.

[3] Rayleigh, Phil. Mag., Vol. 21, 1886, p. 381.

[4] Kelvin, Math. & Phys. Papers, Vol. 3, p. 462.

particular cases of low and high frequency. In a recent paper, Nicholson[1] has published another solution of this problem. He considers first two parallel conductors carrying equal currents in opposite directions and then arrives at the case of the single wire by neglecting the effect of the second upon it. In a paper on the "Effective Resistance and Inductance of a Concentric Main," Russell[2] has given formulas for the virtual resistance and inductance of a concentric main with a solid inner conductor. The expression for the virtual resistance is conveniently arranged so that the resistance of either the core or the sheath may be calculated for any particular case. He shows how the formulas become simplified for the cases of direct current and alternating currents of low and high frequency. A consideration of the current density in the inner and outer conductors of the main is another feature of this paper.

Mordey[3] has employed Kelvin's formulas in the numerical computation of the increase of resistance of a cylindrical conductor, when carrying alternating currents of commercial frequency, 80 to 133 complete cycles per second. Thus with a frequency of 80, the resistance of a copper conductor, 2.5 cm. in diameter, increases 17.5 per cent. The same percentage increase is computated for a conductor of 2.24 cm. at 100 cycles per second, and also for a conductor of 1.036 cm. at 133 cycles per second.

Fleming[4] has measured the ratio of the virtual to the direct current resistance of cylindrical wires using alternating currents of frequency somewhat less than half a million per second. For example, with a frequency of 440,000 using a copper wire, no. 14 standard wire gauge, he finds the ratio to be 5.46 by Russell's formula while his measured value is 5.90. He considers this agreement between the values, which differ by about eight per cent., as satisfactory. Values are given below for cylindrical wires which differ by less than one per cent., and for square and rectangular sectioned wires about four per cent. The experimental work on cylindrical wires, cited below, was performed with slightly damped oscillating currents, to determine whether the damping was great enough to make any appreciable difference between the results observed from damped currents and calculated by a formula derived upon the assumption that the currents were undamped. Fleming used undamped oscillations in obtaining his values.

2. Experimental Method.

(*a*) *Production of Currents of High Frequency.*—The two most common methods of producing alternating currents of high frequency are the

[1] Nicholson, Phil. Mag., Vol. 17, 1909, p. 255.
[2] Russell, Phil. Mag., Vol. 17, 1909, p. 524.
[3] Mordey, Electrician, May 31, 1889, p. 94.
[4] Fleming, Electrician, December 17, 1909, p. 381.

singing-arc method and by the discharge of a condenser through an inductive resistance. Both methods were tried but greater success was obtained by the latter, so it was adopted in the experimental work herein recorded. The oscillations of current from a condenser are more or less damped, but the damping was reduced to such a magnitude, in the circuits here described, that the results did not appreciably differ from those which would have been obtained from using undamped currents.

The scheme of connections is given diagramatically in Fig. 1. *I.C.* is a large six-inch induction coil. Its primary is connected to a source of direct-current supply. *I* is a Cunningham mercury-jet interrupter. It has a wide range of frequency of interruption and when in good working order can be depended upon to give a constant root-mean-square current in the secondary of the induction coil. *C* is a condenser the plates of which are separated by sheets of glass which can resist differences of potential of 30,000 volts or more. This condenser, when charged by the induction coil to the point of breaking down the spark-gap *S*, discharges through the variable inductance *L*.

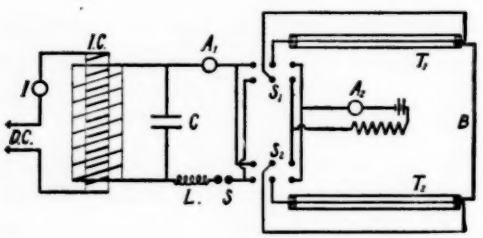

Fig. 1.

Possibly the greatest difficulty in maintaining the discharge of a condenser with sufficient constancy of current, to enable one to use it for precise measurements, lies in the spark-gap used. A long series of experiments showed conclusively that the electrodes must be clean and bright, and separated by an unvarying distance. Zinc electrodes, which are suitable for some purposes, are soon covered by an oxide which causes the width of the gap to vary. Fine jets of mercury were finally selected as the electrodes. The jets were perpendicular to each other in a horizontal plane and at the point of crossing were about a millimeter apart and half a millimeter in diameter. Their surfaces remained always clean and bright and the distances between them seemed to be unvarying. With these arrangements oscillating currents were obtained, adjustable in frequency from twenty thousand to ten millions or more cycles per second, and sufficiently uniform for precise measurements.

(*b*) *Measurement of Intensity of Current.*—In measuring the root-mean-square intensity of high frequency currents, it was found that several precautions were necessary. An instrument was first made which consisted of two parallel bars, each seven centimeters long, one centimeter

wide and three millimeters thick. The bars were connected by seven No. 40 B. & S. gauge high resistance wires, placed one centimeter apart. A nickel-iron thermo-couple was soldered to the middle point of each wire and was properly calibrated by direct current. The alternating current leads were connected to the ends of the bars from the same side. It was found that with a direct current flowing through the instrument each resistance wire conducted one seventh of the total current. An alternating current of frequency about 300,000 was sent through the instrument and readings were taken from the thermo-couples. It was found that the ratio of currents in the two extreme wires was about seven to nine, with the wire nearest the leads carrying the most current. This unequal distribution shows the necessity for caution in the constuction of an ammeter that is to be used for alternating currents of high frequency.

To eliminate the danger of unequal current distribution an ammeter was made as is shown in Fig. 2. In place of the two parallel bars in the instrument described above, there were substituted two triangular blocks of copper, P_1 and P_2, with their parallel edges about three centimeters apart. The bases of the triangles measured 4 cm. and the altitudes 10 cm.

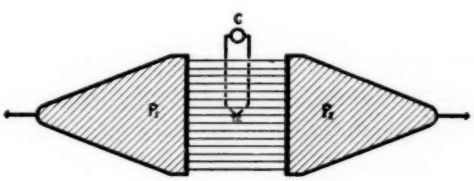

Fig. 2.

The blocks were connected by fifteen no. 40 resistance wires, spaced about 2 mm. apart. On the center wire was soldered a thermo-couple *T.C.*, which is connected to a sensitive D'Arsonval galvanometer *G*. Resistance wires as small as no. 40 were used so that one could be sure that the alternating current resistance, for frequencies up to 1,000,000, was not noticeably different from the direct-current resistance, and hence the instrument could be calibrated by direct current. For currents ranging between one and two amperes, fifteen resistance wires gives about the best sensitiveness. This form of ammeter may be used for a very wide range of current strength by selecting a proper number of resistance wires.

In using the latter instrument to obtain data for this paper each reading of alternating current was calibrated by direct current within a few minutes after the alternating currents had passed through it. This eliminates the danger of incorrect readings caused by a change in room temperature. A wooden box protected the instrument from any air-drafts.

(*c*) *Measurement of Frequency of Alternation.*—Fleming has devised an instrument for measuring the frequency of alternating currents which he calls a photographic spark counter.[1] Since it has been modified in some particulars it will be described here. There is first an enclosing box about 25 cm. square and 40 cm. high. See Fig. 3, which is a diagram showing the plan of the essential parts. A vertical shaft near the middle of one side has mounted on it a four-sided mirror, also a fly wheel to insure uniformity of rotation. A collimator tube projects from one side of the box in line with the mirror. The spark-gap is placed directly in front of the collimator tube. The light from the spark passes through a lens to the mirror and is there reflected at an angle of about ninety degrees to a slit in the side of the box. A suitable plateholder is held in front of the slit and is so constructed that the photographic plate may be raised or lowered by a string passing through the top of the holder. Ordinarily the plate was about twenty centimeters from the mirror. For the higher frequencies it was found necessary to extend the box so that the plate was about fifty centimeters from the mirror. The mirror was rotated by a motor driven by storage cells. The speed of rotation of the mirror could be made as much as one hundred revolutions per second, but was ordinarily rotated at a speed from sixty to eighty revolutions per second. A speed counter attached directly to the upper end of the mirror shaft, with the aid of a stop watch, gave a measure of the angular velocity of the mirror.

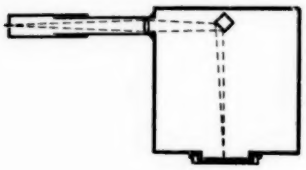

Fig. 3.

The plate was raised or lowered by hand since it was not necessary to know the velocity with which it moved. The projection of the train of oscillations on the plate could easily be made less than the width of the plate by properly modifying the speed of the motor. The sigma brand of Lumière plates was used with excellent results. The average distance between images of sparks was measured by means of calipers and standard scale. From the average distance between images, together with the dimensions of the apparatus involved, the frequency of the sparks was calculated. The accuracy of these measurements is probably within two or three per cent.

In early trials with the apparatus, it was found that the upper limit of frequency of sparks easily counted was about one million per second. At frequencies higher than this, the successive images of individual sparks were fused together into bands, thirty to fifty or sixty in number, each band representing a complete oscillatory discharge of the condenser, and

[1] *Loc. cit.*

the number of bands representing the number of discharges occurring in the well-known multiple manner. With lower frequencies the bands were drawn out into trains of oscillatory sparks. There were usually from twelve to eighteen complete oscillations in each train visible on the plate. Since the intensity of luminosity of the spark becomes gradually weaker, it is impossible to say how many oscillations there are in each train, for the sensitiveness of the plate limits the number of those that can be counted. The character of the photographs has been shown by a number of previous writers, among whom Trowbridge[1] has done extensive work.

(*d*) *The Differential Electric Thermometer.*—To compare the virtual resistance with the direct current resistance, a differential electric thermometer was used. This instrument was devised and used by Fleming.[2] As shown in Fig. 1 it consists of two glass tubes, T_1 and T_2, each about 125 cm. long and 4 cm. in diameter. They are as nearly alike as possible. A small tube with a bore about a millimeter in diameter connects the two larger tubes which are otherwise air-tight. The wires under test pass through rubber stoppers at the ends and complete the electric circuit through either S_1 or S_2, which are highly insulated, six-pole switches. These are used to connect the wire in either tube with the direct current or the oscillating current as may be desired. A small drop of ether or some other light liquid in the capillary tube will indicate any differences of heat generation in the tubes T. It is possible to balance the heat developed by an alternating current in one tube, by an equal amount of heat in the other tube. Since the heat produced in any circuit is proportional to the resistance and the square of the current, at the point of balance the ratio of resistances may be expressed as the inverse ratio of the squares of the currents. From the ammeters in the direct-current circuit and the alternating-current circuit the values of the two currents are read, and from these two readings the ratio of resistances is calculated. In order to minimize any error which might enter into the measurements, because of differences of specific heats of the glass in the two tubes, or in the amount of glass, or radiating power, the currents in the wires were interchanged a number of times and the average results taken.

It was necessary to arrange both the alternating-current circuits, one passing through T_1 and the other through T_2, so that in interchanging circuits during the course of a particular set of readings, there would be no difference of frequency. This was done by changing the area enclosed by one circuit until the frequency of the current in that circuit, as

[1] Trowbridge, Phil. Mag., August, 1894, p. 182.

[2] *Loc. cit.*

measured by means of the photographic plate, was the same as the frequency of the current when passing through the other circuit.

In taking any set of readings for a definite frequency, the method of procedure was to balance three or four different values of alternating current by direct current, with each value of alternating current passing through both circuits T_1 and T_2. The average of these six or eight readings was used to calculate the ratio of R' to R.

The oscillating spark was photographed by the spark counter during one of these readings. The average distance between images on the photographic plate was taken from three or four trains of images and this average distance used in calculating the frequency. The angular velocity of the mirror could be measured within two or three per cent. The frequency of the oscillations could probably be calculated within three or four per cent. of the correct value. The error in the measured ratio of R' to R is not more than four per cent. of the correct value.

(*e*) *Test of the Apparatus.*—In order to test the efficiency of the apparatus, cylindrical wires were introduced into the differential thermometer and observations made on the change of resistance. Another and more important reason for taking this preliminary set of readings, was to discover whether the damping of the oscillations was great enough to cause any appreciable difference between the observed ratio of resistance and that calculated by a formula based upon the assumption that the oscillations were undamped. The results are shown in Table I. Since the experimental readings are within less than one per cent. of the values calculated by Maxwell's formula, it may be assumed that the damping is small enough to neglect, and that the apparatus may be depended upon to give reliable readings, to the degree of precision indicated.

TABLE I.

Variation of Resistance of Cylindrical Wires. Copper Wire, No. 25 Brown and Sharpe Gauge, 0.045 cm. in Diameter.

Frequency.	R'/R Observed.	R'/R Calculated.
36,100	1.000	1.001
138,000	1.062	1.061
283,000	1.215	1.213
313,000	1.242	1.244

Maxwell's formula used in calculating R'/R in the above table is as follows:

$$\frac{R'}{R} = 1 + \frac{\pi^2 N^2 d^4}{48.\rho^2} - \frac{\pi^4 N^4 d^8}{2880.\rho^4} + \cdots,$$

where N is the frequency, d is the diameter of the wire and ρ is the specific resistance. For the wire used above $d = 0.045$ cm. and ρ was taken at 1,600.

To illustrate how the observed values of R'/R were obtained the following data are quoted for the case where the frequency was 283,000.

At the point of balance in the differential electric thermometer:

Avg. direct current through T_1 = 1.725 amps.
Avg. alt. " " T_2 = 1.535 "
$$R'/R = \left(\frac{\text{D.C.}}{\text{A.C.}}\right)^2 = 1.26,$$
Avg. direct " " T_2 = 1.670 "
Avg. alt. " " T_1 = 1.546 "
$$R'/R = \left(\frac{\text{D.C.}}{\text{A.C.}}\right)^2 = 1.17,$$
$$\text{Avg. } R'/R = 1.215.$$

The frequency was obtained from the following formula:

$$N = 4\pi\omega r/f = 283{,}000.$$

where ω, the speed of the mirror, = 70.4 revolutions per second,

r, the distance from the mirror to the plate, = 53.5 cm. and

f, the average distance between images on the plate, = 0.167 cm.

(*f*) *The Virtual Resistance of Wires of Square and Rectangular Cross-Section.*—In applying the formulas no. 31 and no. 27 (developed below) to the experimental results given in Fig. 4, it will be assumed that the effects of damping of the oscillating current are negligible. Formula (27) is more useful for calculating the ratio of apparent resistance to that offered to direct current, in the case of alternating currents of lower frequency.

The calculated values of the ratio of the virtual resistance to the

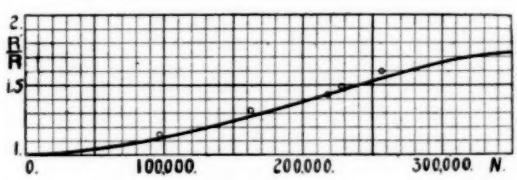

Fig. 4.

direct-current resistance for the wire of which $2a$ = 0.059 cm., were plotted as shown in Fig. 4. The curve is drawn from the calculated values and the small circles indicate the experimental observations.

Fig. 5 shows in a similar manner the nearness with which the experi-

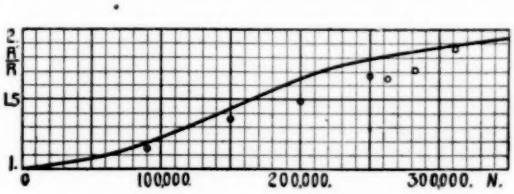

Fig. 5.

mental values check the calculated ones for the square-sectioned wire of which $2a = 0.070$ cm.

Replacing the square wires by wires of rectangular section, another series of observations was obtained, the results of which are shown in Fig. 6 and Fig. 7. The calculated values of R'/R for the wire of which

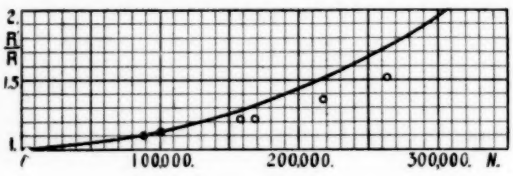

Fig. 6.

$2a = 0.039$ cm. and $2b = 0.0665$ cm., were used in drawing the curve of Fig. 6, the small circles indicating, as above, the experimental readings.

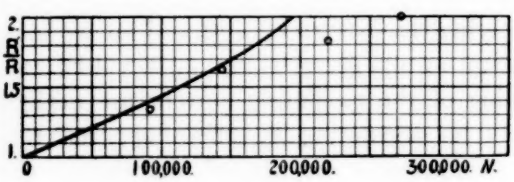

Fig. 7.

Similarly Fig. 7 was drawn for the wire of which $2a = 0.0485$ cm. and $2b = 0.127$ cm. With a wire as large as this last one, the formula does not give values which agree satisfactorily with the experimental values if the frequency is much greater than 150,000. Formulas (26) and (30) were used here in calculating the values for R'/R (see below).

3. Mathematical Development.

The conductor, rectangular in cross-section, is so long that end effects may be neglected. Other conductors are so far removed that their influence need not be considered. Referring to Fig. 8, a reference system is chosen, with the origin at the center of any cross-section, the X and Y axes parallel to the sides of the section, and the Z axis coinciding with the axis of the wire. Since the conductor is long, it may be assumed that no component of current flows parallel to the XY plane. The following equation then holds:

$$-4\pi\mu w = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}, \qquad (1)$$

where, in agreement with Maxwell's notation, w is the z component of current intensity, H is the z component of vector potential of magnetic induction and μ is the permeability of the material of the conductor.

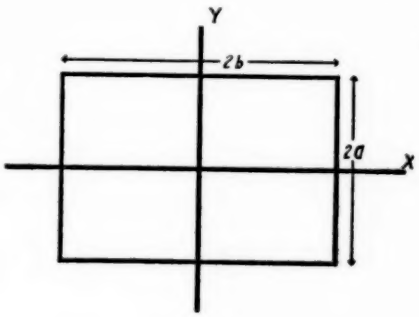

Fig. 8.

The impressed electromotive intensity may be supposed equal at all points of the cross-section of the wire, assuming that the specific resistance is uniform. Let the electromotive force per unit length be $E\epsilon^{int}$. Considering only instantaneous relations, the total electromotive intensity, inductive and non-inductive, in the wire is

$$P = E - \frac{\partial H}{\partial t}. \tag{2}$$

Outside the wire the corresponding electromotive intensity is

$$P = -\frac{\partial H}{\partial t}. \tag{3}$$

If the specific resistance of the wire is ρ and the specific inductive capacity of the medium is K

$$w = \left(\frac{1}{\rho} + \frac{inK}{4\pi}\right) P. \tag{4}$$

Considering the inductive action only,

$$P = -\frac{\partial H}{\partial t}, \tag{5}$$

and eliminating w between equations (1) and (4)

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} - 4\pi\mu in\left(\frac{1}{\rho} + \frac{inK}{4\pi}\right) H = 0. \tag{6}$$

In the wire the current may be regarded as due to the conductivity alone, and if

$$k^2 = 4\pi\mu in/\rho$$

then

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} - k^2 H = 0. \tag{7}$$

In the surrounding medium $1/\rho$ may be neglected and putting $h = n/V$,

where V, the velocity of propagation of the disturbance, is $1/\sqrt{K\mu}$,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + h^2 H = 0. \tag{8}$$

The general solution of (7) is

$$H = A\epsilon^{kx} + B\epsilon^{-kx} + C\epsilon^{ky} + D\epsilon^{-ky} + F\epsilon^{k'x+y} + M\epsilon^{-k'x+y} + N\epsilon^{k'x-y} + Q\epsilon^{-k'x-y}, \tag{9}$$

where $k' = k/\sqrt{2}$, and if $h' = h/\sqrt{2}$ the solution of (8) is

$$H = A'\epsilon^{ihx} + B'\epsilon^{-ihx} + C'\epsilon^{ihy} + D'\epsilon^{-ihy} + F'\epsilon^{ih'x+y} + M'\epsilon^{-ih'x+y} + N'\epsilon^{ih'x-y} + Q'\epsilon^{-ih'x-y}. \tag{10}$$

By symmetry relations the value of H must remain unchanged if $+x$ is written for $-x$ and $+y$ for $-y$ in the equations (9) and (10). This shows at once that $A = B$, $A' = B'$, $C = D$, $C' = D'$, $F = M = N = Q$ and $F' = M' = N' = Q'$. Hence inside the wire

$$H = 2A\cosh kx + 2C\cosh ky + 2F(\cosh k'\overline{x+y} + \cosh k'\overline{x-y}), \tag{11}$$

and outside

$$H = 2A'\cosh ihx + 2C'\cosh ihy + 2F'(\cosh ih'\overline{x+y} + \cosh ih'\overline{x-y}). \tag{12}$$

If the wire has a square cross-section instead of a rectangular section then x can be interchanged with y in (11) and (12) without affecting H. This necessitates the additional simplification that $A = C$ and $A' = C'$.

To evaluate the constants of (11) and (12), the continuity of the total electromotive intensity and magnetic field intensity, at the boundary, is used. If α and β are the x and y components of magnetic field intensity, then since the curl of the vector potential is the magnetic field intensity, at the point $x = b$, $y = 0$, by the continuity of β the following relation is obtained from (11) and (12):

$$kA\sinh kb + 2k'F\sinh k'b = ihA'\sinh ihb + 2ih'F'\sinh ih'b. \tag{13}$$

Similarly by the continuity of α at $x = 0$, $y = a$,

$$kC\sinh ka + 2k'F\sinh k'a = ihC'\sinh iha + 2ih'F'\sinh ih'a. \tag{14}$$

Also at the point $x = b$, $y = a$, for both α and β,

$$kA\sinh kb + k'F(\sinh k'\overline{b+a} + \sinh k'\overline{b-a}) = ihA'\sinh ihb + ih'F'(\sinh ih'\overline{b+a} + \sinh ih'\overline{b-a}), \tag{15}$$

$$kC\sinh ka + k'F(\sinh k'\overline{b+a} - \sinh k'\overline{b-a}) = ihC'\sinh ih'a + ih'F'(\sinh ih'\overline{b+a} - \sinh ih'\overline{b-a}). \tag{16}$$

Also by the continuity of the total electromotive intensity at $x = b$ $y = 0$ and at $x = 0$, $y = a$ using equations (2) and (3), two more relations may be obtained:

$$E - 2inA \cosh kb - 4inF \cosh k'b = -2inA' \cosh ihb - 2inF' \cosh ih'b, \quad (17)$$

$$E - 2inC \cosh ka - 4inF \cosh k'a = -2inC' \cosh iha - 2inF' \cosh ih'a. \quad (18)$$

From equations (13) to (18) the values of the constants may be found.

The total current passing through any cross-section may be found by taking the line integral of the magnetic field intensity around the boundary. If γ is the total current

$$\gamma = 4\int_0^a \beta dy - 4\int_0^b \alpha dx. \quad (19)$$

Inserting the values of α and β in this equation and integrating gives the desired electromotive force equation, from which, expressions for the virtual resistance and inductance may be calculated. The determinations of the constants by equations (13) to (18) is so complicated as to be unmanageable. It is possible, however, to obtain approximate formulas with mean value for the constants by assuming that $A = C = F$ in equation (11) and neglecting the effect of the medium. The assumption that $A = C$ means that the vector potential along the x axis is the same as along the y axis for equal values of the argument. This is true for a square sectioned wire but when considering a rectangular sectioned conductor it is only approximately true, the degree of approximation depending upon the difference between the two dimensions. The justification for putting $A = F$ is based upon the substantiation of the results as calculated for particular cases from experimental observations. The values of R'/R for four particular cases, calculated by a formula based upon this assumption are given in Figs. 5–8. Since the observed values agree up to frequencies of 150,000, with these calculated values, it must be that up to this limit F is nearly equal to A. For the two rectangular sectioned wires this limit is reached at frequencies of 150,000 and then the formulas fail, then the differences between the observed and calculated values increases as the frequency increases. For the larger square sectioned wire the values agree within five per cent. up to a frequency of 150,000, and then the differences increase to about ten per cent. at a frequency of 265,000, and at 312,000 the values are again in close agreement. The differences for the smaller square sectioned wire are not greater than three per cent. for the range tested, namely up to a frequency of 257,000.

To obtain these approximate expressions equation (7) will be used, but put into more convenient form. Introducing the non-inductive part of the electromotive intensity, which may be called $\partial\psi/\partial z$, into equation (4) and neglecting the effect of the medium, gives

$$\rho w = -\frac{\partial\psi}{\partial z} - \frac{\partial H}{\partial t}. \tag{20}$$

Eliminating w between this equation and equation (1)

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} - k^2\left(H - \frac{i}{n}\frac{\partial\psi}{\partial z}\right) = 0. \tag{21}$$

Since $\frac{i}{n}\frac{\partial\psi}{\partial z}$ is independent of x or y, (21) may be written

$$\frac{\partial^2\left(H - \frac{i}{n}\frac{\partial\psi}{\partial z}\right)}{\partial x^2} + \frac{\partial^2\left(H - \frac{i}{n}\frac{\partial\psi}{\partial z}\right)}{\partial y^2} - k^2\left(H - \frac{i}{n}\frac{\partial\psi}{\partial z}\right) = 0. \tag{22}$$

The general solution of this equation is similar to the solution of (7) and has eight undetermined coefficients. This number may be reduced to three by the symmetry relations as is shown above. Assuming now that these three constants are all equal leads to the following solution:

$$H = \frac{i}{n}\frac{\partial\psi}{\partial z} + 2A(\cosh kx + \cosh ky + \cosh k'\overline{x+y} + \cosh k'\overline{x-y}). \tag{23}$$

The value of the single undetermined coefficient may be found by integrating the expression for current intensity over the surface of the cross-section. If γ is the total current,

$$\gamma = 4\int_0^a\int_0^b w\,dy\,dx.$$

Using the value of w as determined by equations (1) and (23), with the aid of the above integral, the value of A may be shown to be

$$A = -\frac{\pi\mu\gamma}{2(ka\sinh kb + kb\sinh ka + 2\cosh k'\overline{b+a} - 2\cosh k'\overline{b-a})}. \tag{24}$$

If at the point $x = b$, $y = 0$, H is put equal to some constant, say L, times the current and if l be the length of the conductor considered, multiplying equation (23) by lin and putting $-l\frac{\partial\psi}{\partial z} = E$, the electromotive force, gives

$$E = Llin\gamma$$

$$+ lin\pi\mu\gamma\frac{\cosh kb + 2\cosh k'b + 1}{ka\sinh kb + kb\sinh ka + 2\cosh k'\overline{b+a} - 2\cosh k'\overline{b-a}}. \tag{25}$$

If the hyperbolic functions are expanded in series of powers of k and the numerator divided by the denominator, the following expression is obtained:

$$E = lin\gamma\left[L + \frac{\pi\mu(2b^2-a^2)}{12ab} - \frac{\pi^5\mu^3N^2(-b^6+140b^4a^2+56b^2a^4-29a^6)}{3780\cdot ab\rho^2} + \cdots\right]$$
$$+ R\gamma\left[1 + \frac{\pi^4\mu^2N^2(4b^4+30b^2a^2-11a^4)}{45\rho^2}\right. \tag{26}$$
$$\left. - \frac{\pi^8\mu^4N^4(-172b^8+1070b^6a^2+2807b^4a^4-340b^2a^6-142a^8)}{14175\cdot\rho^4} + \cdots\right].$$

where $R = l\rho/4ab$ (the resistance of a length l for direct current) and $N = n/2\pi$, the number of alternations per second. Writing R' for the virtual resistance, gives

$$\frac{R'}{R} = 1 + \frac{\pi^4\mu^2N^2(4b^4+30b^2a^2-11a^4)}{45\rho^2} - \cdots. \tag{27}$$

If the wire has a square cross-section the expression for the virtual resistance may be found by putting $b=a$ in (26)

$$\frac{R'}{R} = 1 + \frac{23\pi^4\mu^2N^2a^4}{45\rho^2} - \frac{3223\pi^8\mu^4N^4a^8}{14175\cdot\rho^4} + \cdots. \tag{27}$$

Formulas (26) and (27) may be used in calculating R'/R for particular cases where the dimensions and frequency are of a magnitude which will give a value of 0.5 or less to the second term of the series. If the frequency and dimensions are large then the series is not rapidly convergent. The calculation of more terms of the series is troublesome. To obtain a complete expression the fraction in (25) may be expressed in exponential functions and then separated into real and imaginary parts. The real coefficient of γ is then the desired expression for R'. The device used in making the necessary transformation is embodied in the following relations:

$$k = \sqrt{2i}\sqrt{\frac{2\pi\mu n}{\rho}} = (1+i)S,$$

where

$$S = \sqrt{\frac{2\pi\mu n}{\rho}}. \tag{28}$$

Hence each hyperbolic function may be expressed in somewhat the following manner:

$$\cosh kb = \tfrac{1}{2}(e^{kb} + e^{-kb}) = \cos bS \cosh bS + i \sin bS \sinh bS.$$

Making these substitutions in (25) and then rationalizing the denominator of the fraction, gives

$$E = Llin\gamma + lin\pi\mu\gamma \frac{(A_1C_1 + B_1D_1) + i(B_1C_1 - A_1D_1)}{C_1^2 + D_1^2}, \qquad (29)$$

where

$$A_1 = \cos bS \cosh bS + 2 \cos \frac{bS}{\sqrt{2}} \cosh \frac{bS}{\sqrt{2}} + 1,$$

$$B_1 = \sin bS \sinh bS + 2 \sin \frac{bS}{\sqrt{2}} \sinh \frac{bS}{\sqrt{2}},$$

$$C_1 = aS \cos bS \sinh bS - aS \sin bS \cosh bS + bS \cos aS \sinh aS$$

$$- bS \sin aS \cosh aS + 2 \cos \frac{b+a}{\sqrt{2}} S \cosh \frac{b+a}{\sqrt{2}} S - 2 \cos \frac{b-a}{\sqrt{2}} S \sinh \frac{b-a}{\sqrt{2}} S,$$

$$D_1 = aS \cos bS \sinh bS + aS \sin bS \cosh bS + bS \cos aS \sinh aS$$

$$+ bS \sin aS \cosh aS + 2 \sin \frac{b+a}{\sqrt{2}} S \sinh \frac{b+a}{\sqrt{2}} S - 2 \sin \frac{b-a}{\sqrt{2}} S \sinh \frac{b-a}{\sqrt{2}} S.$$

Hence:

$$\frac{R'}{R} = \frac{4n\pi\mu ab}{\rho} \times \frac{A_1D_1 - B_1C_1}{C_1^2 + D_1^2}. \qquad (30)$$

For the wire of square cross-section this expression becomes simplified and

$$\frac{R'}{R} = \frac{4\pi n\mu a^2}{\rho} \times \frac{A_2D_2 - B_2C_2}{C_2^2 + D_2^2}, \qquad (31)$$

where the change in subscripts indicate the changes in the constants which are obtained by putting $b = a$.

It is shown above that these expressions can be used to calculate the virtual resistance within certain limits.

Before leaving the mathematical side of the investigation an expression for finding the intensity of current at any point of the cross-section of the conductor is to be developed. For present purposes a study of the distribution of current in a wire of square cross-section is just as instructive as that in a wire of rectangular cross-section.

Using equation (1) for current intensity and (23) for vector potential the following expression, for the wire of square section, is obtained:

$$4w = \frac{\gamma[k^2(\epsilon^{kx} + \epsilon^{-kx} + \epsilon^{ky} + \epsilon^{-ky}) + k'^2(\epsilon^{k'x+y} + \epsilon^{-k'x+y} + \epsilon^{k'x-y} + \epsilon^{-k'x-y})]}{2(\epsilon^{2k'a} + \epsilon^{-2k'a}) + 2ka(\epsilon^{ka} - \epsilon^{-ka}) - 4}. \qquad (32)$$

The exponential functions of this expression may be separated into real and imaginay parts by a method of procedure similar to the method used above.

$$4w = \frac{\gamma S^2[(A_3C_3 - D_3B_3) + i(A_3D_3 + B_3C_3)]}{C_3^2 + D_3^2}, \tag{33}$$

where

$$A_3 = -\left[\sin Sx \sinh Sx + \sin Sy \sinh Sy + \sin \frac{x+y}{\sqrt{2}} S \sinh \frac{x+y}{\sqrt{2}} S + \sin \frac{x-y}{\sqrt{2}} S \sinh \frac{x-y}{\sqrt{2}} S\right],$$

$$B_3 = \cos Sx \cosh Sx + \cos Sy \cosh Sy + \cos \frac{x+y}{\sqrt{2}} S \cosh \frac{x+y}{\sqrt{2}} S + \cos \frac{x-y}{\sqrt{2}} S \cosh \frac{x-y}{\sqrt{2}} S,$$

$$C_3 = \cos \sqrt{2}\, aS \cosh \sqrt{2}\, aS + aS \cosh aS \sinh aS - aS \sin aS \cosh aS - 2,$$

$$D_3 = \sin \sqrt{2}\, aS \sinh \sqrt{2}\, aS + aS \cos aS \sinh aS + aS \sin aS \cosh aS.$$

The equation (33) expressing current intensity at any point in the cross-section of the wire is like the general electromotive force equation in that it is a complex quantity. The form must be complex because part of the current is caused by the inductive action and the other part by the non-inductive action. If equation (33) were multiplied by ρ, the specific resistance, the resulting equation would be an electromotive force equation, which is in general complex.

To obtain values of w at points in the cross-section of a particular wire under given conditions, the real and imaginary parts must be calculated separately, and then added by vector methods, or analytically by finding the square root of the sum of their squares.

4. Variation of Current Intensity.

Another purpose of this paper is to show the variation of current intensity or density of flow, over the sectional area of a long, straight conductor of square cross-section, carrying current of high frequency. The mathematical development leads to a computation of current intensity at any point of the section. Since no experimental method of measuring this quantity has thus far been devised, the equations derived from theoretical considerations will be relied upon entirely. The justification for this course lies in the satisfactory agreement between experimental and theoretical results, already described in another part of this paper. It appears that the experimental difficulties in the measurement of current

intensity would be considerable. If the wire be separated into filaments, like the fine, parallel wires of a flexible cable, the conditions which cause the uneven distribution of current have been modified to such an extent that the measurement of current in each filament would be of little value in solving the problems proposed. Equation (33) will be used in the calculations which follow.

A wire or bar, two centimeters square, has been selected for illustration. To simplify the calculation let the frequency be $400/\pi^2$. This makes S of equation (33) unity. Since the wire is large in cross-section, even with low frequency the variation in current intensity will be similar to the variation in a smaller wire with higher frequency. The similarity may be seen from an inspection of formula (32). It will be noticed that wherever $\sqrt{N}$ appears in the equation as a factor, a dimensional length x, y or a appears also as a factor. N appears elsewhere only in K^2. If $\sqrt{N}d$ (where d is a dimension) is put equal to some constant, then $\sqrt{N}$ and d may be varied individually, so long as their product remains constant, and the resulting value for w will be proportional to the frequency.

Consider the square section as a special case of the rectangular section of Fig. 8 in which b equals a. Fig. 9 shows graphically the variation of current intensity along the three principal lines of the particular square selected. Curve A represents the variation from the origin along the line $y = 0$ to $x = \pm a$. Curve B shows the variation from the middle of any side on either axis along the edge to the vertex. This curve represents the symmetrical variation of current along eight different lines of the square. In curve C the variation of current intensity along a diagonal from the origin to any one of the four vertices is shown.

Another graphical representation of the density of current flow over the section of the same square wire, at the same frequency of alternation, is made in Fig. 10. Here contour lines are drawn for one quadrant of the section, the difference between consecutive lines being 0.005 C.G.S. unit of current. The line nearest the origin indicates an intensity of 0.215 C.G.S. unit. The current is fairly uniform over the central region, but proceeding from the origin along a diagonal to a vertex the lines are drawn more closely together. It might be supposed at first sight that for the case of very high frequency the current would be localized entirely at the corners. That such is not the case may be seen by inspection of formula (33). Putting x and y both equal to zero, the numerator still contains N, which, although small in comparison with the denominator, gives w a value different from zero.

In any electromagnetic phenomenon the law of the conservation of energy is valid. In circuits such as described above the energy is mani-

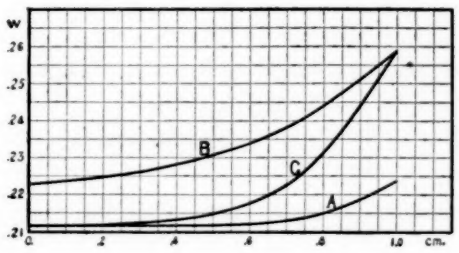

Fig. 9.

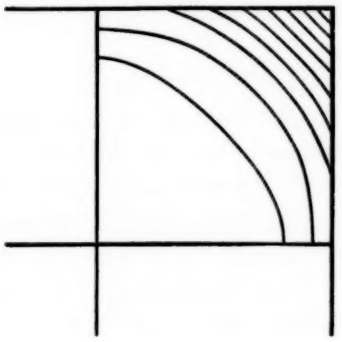

Fig. 10.

fested in two ways, namely, in the production of heat and in the production of the magnetic field. There would be a minimum of heat produced if the current were distributed uniformly over the cross-section. The energy of the magnetic field would be a minimum if the current were entirely on the surface of the wire. Since both of these conditions cannot be satisfied simultaneously, a compromise is established which makes the total energy a minimum. This compromise is shown for a particular case in Fig. 11.

To show the variation in current intensity for different frequencies, also to show that the variation in a small wire with high frequency is similar to the variation in a large wire with low frequency, the values of w for a wire having $2a = 0.070$ cm., with a frequency of 101,320, have been calculated along the line corresponding to curve A of Fig. 9 and are shown in Fig. 11. The increase in intensity from the smallest to the greatest value in the smaller wire is greater than in the larger wire, but the characteristics of the two curves are the same. A value of the frequency could be found that would make the increase the same in both wires.

The effect of increasing the frequency of alternation of current in the same wire of square section is illustrated by the two curves of Fig. 11. Both curves are for the line $y = 0$ from the origin to $x = a$. The length of one side of the section is $2a = 0.070$ cm. Curve A shows the variation for a frequency of 101,320, and curve B for a frequency of

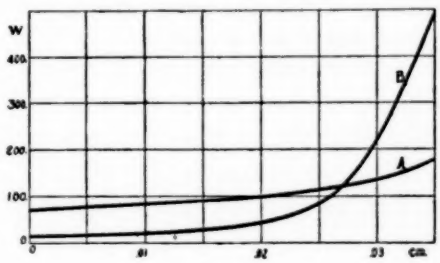

Fig. 11.

1,000,000. The wire is supposed to carry the same total current in both cases, ten amperes. The effect of increasing the frequency is shown in a crowding of the current toward the boundaries of the wire.

SUMMARY.

1. Alternating currents have been produced by the discharge of a condenser through an inductive resistance, with sufficient uniformity of current to render possible their use in precise measurements.

2. An ammeter is described which is suitable for the measurement of the intensity of alternating currents.

3. The frequency of alternating currents has been measured by a photographic spark counter for any frequency up to three quarters of a million per second.

4. By checking Maxwell's formula for the virtual resistance of cylindrical wires, it is shown that the damping of the oscillating currents used is small enough to neglect.

5. The ratio of the virtual to the direct current resistance for copper wires of square and rectangular cross-section was measured by the use of a differential thermometer, and these observed values were checked by approximate formulas within certain limits. For the two square cross-sectioned wires tested this limit was for frequencies above one hundred and fifty thousand and for the two rectangular cross-sectioned wires the limit was about one hundred and fifty thousand.

6. Formulas are developed by certain assumptions in the general theory, which give approximate methods of calculating the ratio of the virtual to the direct current resistance. The validity of these assumptions is based upon experimental results.

7. In the development of the expression from which the ratio of resistances is calculated, there occurs an expression for current intensity at any point of the cross-section of the conductor. This expression is used to calculate, for some particular cases, the distribution of current intensity within the cross-section.

UNIVERSITY OF CALIFORNIA,
May 1, 1911.

ON THE NOMENCLATURE OF CRYSTALLOGRAPHY.

By Paul Saurel.

THE thirty-two types of crystal symmetry have received so many different names that it may seem useless to add to the reigning confusion by offering a new set. And yet the nomenclature and the symbols which I wish to suggest recall so simply the characteristic elements of symmetry of each of the thirty-two types that I cannot refrain from submitting them to the judgment of physicists.

But first it will be necessary to recall briefly the various possible types of symmetry. It is well known that the only elements of symmetry that need be considered are a center of symmetry, axes of direct symmetry and axes of inverse symmetry.[1] The point O is said to be a center of a symmetry of a system if to every point P of the system there corresponds a point P' of the system such that the line PP' is bisected by the point O. The points P and P' are said to be inverses of each other with respect to the center O, and instead of saying that the system possesses a center of symmetry we may say that inversion with respect to O transforms the system into itself. A line OA is said to be an axis of direct symmetry of order n if rotation of the system about OA through an angle equal to $1/n$th of an entire revolution causes every point of the system to take a position previously occupied by a point of the system. Thus we may say that rotation about an axis of direct symmetry transforms the system into itself. Finally, a line OA is said to be an axis of inverse symmetry of order n if rotation about OA through an angle equal to $1/n$th of an entire revolution followed by an inversion with respect to O transforms the system into itself.

The various possible types of symmetry fall into four groups: (1) the types that are characterized by the existence of axes of direct symmetry alone; (2) the types that are characterized by the presence of a center of symmetry; (3) the types that are characterized by the presence of axes of inverse symmetry without a center of symmetry; (4) the type that is characterized by the absence of all elements of symmetry.

The types of symmetry that are characterized by the existence of

[1] For demonstrations of the various statements made in the text the reader is referred to H. A. Lorentz, "Über die Symmetrie der Kristalle"; Abhandlungen über Theoretische Physik, Vol. 1, 1907, p. 299, and to Paul Saurel, "On the Classification of Crystals," Bulletin of the American Mathematical Society, Vol. 17, 1911, p. 398.

axes of direct symmetry alone are five in number which, following the usage of mathematicians, we shall call the cyclic type, the dihedral type, the tetrahedral type, the octahedral type and the icosahedral type.

The cyclic type of symmetry is characterized by the presence of a single axis of direct symmetry of order n, where n is any integer except 1. We shall use the symbol C_n to represent this type of symmetry. Moreover, we shall agree to use the symbol C_1 to denote the absence of all symmetry.

The dihedral type of symmetry is characterized by the axes of symmetry which a regular polygon of n sides possesses. These consist of an n-ary axis perpendicular to the plane of the polygon and passing through its center, a set of n binary axes consisting of the radii drawn from the center of the polygon to its vertices, and a second set of n binary axes consisting of the radii drawn from the center of the polygon to the mid-points of its sides. The above general statement is to be understood to include the case $n = 2$; for this purpose it is necessary to agree to use the expression regular polygon of two sides to denote a limited straight line lying in a given plane. The axes of symmetry in this case consist of a binary axis perpendicular to the given plane and passing through the mid-point of the line, a pair of binary axes drawn from the mid-point of the line to its ends and a pair of binary axes lying in the given plane and drawn perpendicular to the given line from its mid-point. We shall use the symbol $D_{2,\,2,\,n}$ to denote the dihedral type of symmetry; the subscripts recall the orders of the axes.

The tetrahedral type of symmetry is characterized by the axes of symmetry which a regular tetrahedron possesses, namely, four ternary axes connecting the vertices of the tetrahedron with the mid-points of the opposite faces and three binary axes connecting the mid-points of opposite edges. We shall use the symbol $T_{2,\,3,\,3}$ to represent this type of symmetry.

The octahedral type of symmetry is characterized by the axes of symmetry which a regular octahedron possesses, namely, three quaternary axes connecting opposite vertices, four ternary axes connecting the mid-points of opposite faces, and six binary axes connecting the mid-points of opposite edges. We may also say that this type of symmetry is characterized by the axes of symmetry which a cube possesses, namely, three quaternary axes connecting the mid-points of opposite faces, four ternary axes connecting opposite vertices, and six binary axes connecting the mid-points of opposite edges. We shall use the symbol $O_{2,\,3,\,4}$ to represent this type of symmetry; the subscripts recall the orders of the axes.

The icosahedral type of symmetry is characterized by the axes of symmetry which a regular icosahedron possesses, namely, six quinary axes connecting opposite vertices, ten ternary axes connecting the mid-points of opposite faces, and fifteen binary axes connecting the mid-points of opposite edges. We may also say that this type of symmetry is characterized by the axes of symmetry which a regular dodecahedron possesses, namely, six quinary axes connecting the mid-points of opposite faces, ten ternary axes connecting opposite vertices, and fifteen binary axes connecting the mid-points of opposite edges. We shall use the symbol $I_{2,3,5}$ to represent this type of symmetry; the subscripts recall the orders of the axes.

From the types of symmetry C_n, $D_{2,2,n}$, $T_{2,3,3}$, $O_{2,3,4}$, $I_{2,3,5}$, characterized by the presence of axes of direct symmetry alone, it is easy to obtain the types of symmetry that are characterized by the presence of a center of symmetry. It is sufficient, indeed, to add to the axes of symmetry of each type a center of symmetry. We thus obtain five new types of symmetry which we shall denote by the symbols $\overline{C}_n$, $\overline{D}_{2,2,n}$, $\overline{T}_{2,3,3}$, $\overline{O}_{2,3,4}$, $\overline{I}_{2,3,5}$, and which we shall call the centro-cyclic, the centro-dihedral, the centro-tetrahedral, the centro-octahedral and the centro-icosahedral types; the bar will serve to recall the existence of a center of symmetry.

Finally from the types of symmetry characterized by direct axes of symmetry it is easy to obtain the types of symmetry characterized by the presence of inverse axes of symmetry. In the first place, from the cyclic type of even order C_{2n}, we obtain a new type of symmetry by replacing the direct axis of order $2n$ by an inverse axis of the same order. We shall denote this type by the symbol $C_{\overline{2n}}$, and we shall call it the inverse cyclic type. In the second place, from the dihedral type $D_{2,2,n}$, and the octahedral type $O_{2,3,4}$ we obtain new types of symmetry by replacing any two sets of direct axes of even order by inverse axes of the same orders. We shall denote these new types by the symbols $D_{\overline{2},\overline{2},n}$, $D_{2,\overline{2},\overline{2n}}$ where $n>1$, $O_{\overline{2},3,\overline{4}}$ and we shall call them the inverse dihedral types of the first and second kind, and the inverse octahedral type. The bars, of course, serve to recall the axes of inverse symmetry.

The various types of symmetry are summarized in the following table.

Types of Symmetry characterized by

Axes of Direct Symmetry	Center of Symmetry	Axes of Inverse Symmetry	
C_n	$\overline{C}_n$	$C_{\overline{2n}}$	$n \geqq 1$
$D_{2,2,n}$	$\overline{D}_{2,2,n}$	$D_{\overline{2},\overline{2},n}$, $D_{2,\overline{2},\overline{2n}}$	$n \geqq 2$
$T_{2,3,3}$	$\overline{T}_{2,3,3}$		
$O_{2,3,4}$	$\overline{O}_{2,3,4}$	$O_{\overline{2},3,\overline{4}}$	
$I_{2,3,5}$	$\overline{I}_{2,3,5}$		

Not all of the types of symmetry enumerated in this table are available as types of crystal symmetry, for the law of rational indices limits the acceptable axes of symmetry to those of orders 1, 2, 3, 4, 6. With this limitation the table furnishes the 32 types of crystal symmetry, 11 from each of the first two columns and 10 from the third. These 32 types of symmetry fall naturally into six groups; the first group consists of the types that correspond to the regular solids, the second group consists of the types that contain a senary axis, while the third, fourth, fifth and sixth groups consist respectively of the types that contain a quaternary, a ternary, a binary or no axis. The following table contains the symbols and the names which are proposed for each of the thirty-two types of crystal symmetry; the names and especially the symbols recall concisely the characteristic elements of symmetry of the various types.

Regular System.

1. $\bar{O}_{2,3,4}$ Centro-octahedral type.
2. $O_{2,3,4}$ Octahedral type.
3. $O^{-}_{,\bar{3},4}$ Inverse octahedral type.
4. $\bar{T}_{2,3,3}$ Centro-tetrahedral type.
5. $T_{2,3,3}$ Tetrahedral type.

Senary System.

1. $\bar{D}_{2,2,6}$ Centro-dihedral type.
2. $D_{2,2,6}$ Dihedral type.
3. $D_{\bar{2},\bar{2},6}$ First inverse dihedral type.
4. $D_{2,\bar{3},\bar{6}}$ Second inverse dihedral type.
5. $\bar{C}_6$ Centro-cyclic type.
6. C_6 Cyclic type.
7. C^{-} Inverse cyclic type.

Quaternary System.

1. $\bar{D}_{2,2,4}$ Centro-dihedral type.
2. $D_{2,2,4}$ Dihedral type.
3. $D_{\bar{2},\bar{2},4}$ First inverse dihedral type.
4. $D_{2,\bar{2},\bar{4}}$ Second inverse dihedral type.
5. $\bar{C}_4$ Centro-cyclic type.
6. C_4 Cyclic type.
7. $C_{\bar{4}}$ Inverse cyclic type.

Ternary System.

1. $\bar{D}_{2,2,3}$ Centro-dihedral type.
2. $D_{2,2,3}$ Dihedral type.
3. $D_{\bar{2},\bar{2},3}$ Inverse dihedral type.
4. $\bar{C}_3$ Centro-cyclic type.
5. C_3 Cyclic type.

Binary System.

1. $\bar{D}_{2,2,2}$ Centro-dihedral type.
2. $D_{2,2,2}$ Dihedral type.
3. $D_{\bar{2},\bar{2},2}$ Inverse dihedral type.
4. $\bar{C}_2$ Centro-cyclic type.
5. C_2 Cyclic type.
6. C^- Inverse cyclic type.

Anaxial System.

1. $\bar{C}_1$ Centric type.
2. C_1 Acentric type.

NEW YORK, April 25, 1911.

ON THE POSITIVE POTENTIAL OF METALS IN THE PHOTOELECTRIC EFFECT AND THE DETERMINATION OF THE WAVE-LENGTH EQUIVALENT OF ROENTGEN RAYS.

BY JAKOB KUNZ.

IT has been stated in two previous papers, published in this journal,[1] that the positive potential acquired by alkali metals under the action of visible light, presents different features in the red end and the blue end of the spectrum. In the red end the positive potential acquires a maximum as function of the frequency and depends on the intensity of the light; towards the violet end, however, the positive potential is largely independent of the intensity of the incident light, and increases proportional to the square of the frequency. When the intensity of the light used becomes very small, then the positive potential apparently depends on the intensity, but this is very likely due to the fact that the insulation is not perfect, so that the positive potential measured by the electrometer, never rises to that constant value which it acquires if the number of electrons emitted by the metal is larger. Nevertheless it is necessary to find out by means of experiments whether the positive potential in the ultra-violet region increases continuously with the square of the frequency or whether there is a second maximum similar to that in the red end of the spectrum. If this be the case, then we are dealing with phenomena of optical resonance alone while in the other case, where the positive potential increases with the square of the frequency, the phenomenon would be of a different character and would give an exact means of measuring the thickness of the pulse or the wave length equivalent of Roentgen rays, γ-rays and secondary Roentgen radiation.

Two different methods of measuring the positive potentials have been used. The first method was described in the previous papers, the positive potential being measured by means of an electrometer of high sensitiveness. That part of the tube which contained the alkali metal consisted of a quartz tube, or of a glass tube of the shape of Fig. 1, covered with a plate of quartz *C*. The quartz plate was sealed to the ground glass tube by means of Kotinsky cement, so that a very good vacuum could be

[1] PHYS. REV., Vol. XXIX., No. 3, September, 1909, p. 227; Vol. XXXI., No. 5, November, 1910, p. 538.

obtained and increased by means of charcoal and liquid air. The alkali metal was introduced into D by means of a system of tubes and funnels as described before. It was found to be very important to use a perfectly clean metallic surface as a trace of impurity on the surface decreased the positive potential considerably or even annulled it. Thus the funnels had an opening of about 0.6 mm. diameter. An electrode made connection between the positive metal and the electrometer. Two negative electrodes A and B were used. A, a fine wire net blackened with lamp black, was introduced in order to establish an auxiliary potential difference between A and the aluminum electrode B, through which reflection of electrons from B might be prevented. The incidence of the ultra-violet light was perpendicular so as to prevent the reflected light from falling on the negative electrode B, and producing a positive potential.

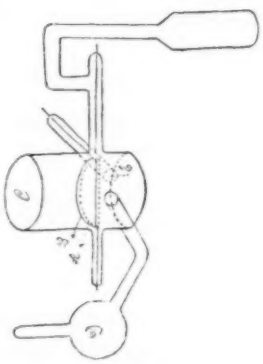

Fig. 1.

The ultra-violet light was produced by the discharge from a 3-kilowatt transformer between cadmium zinc and copper terminals in parallel with a suitable capacity, of three or four Leyden jars. The light passed through a specially prepared spectroscope, quartz lenses and a Cornu prism. The light incident upon the metal was always focused on the surface by a displacement of the last lens in the spectroscope.

The metals cæsium and rubidium were prepared by the following method: 14 gr. cæsium chloride molten in a stream of dry hydrochloric acid gas are mixed after cooling with 2.5 gr. commerical calcium, placed in an iron boat and in a tube of Bohemian glass; a plug of clean asbestos and iron wire keep the boat in a given position and prevent the mixture from spreading together with the metal vapor to the lower part of the glass tube, which is connected by means of glass tubes and sealing wax with the system of tubes, of Fig. 1. The glass tube is gradually heated to bright red heat, where the cæsium distils off and flows down into the receptacle. The distillation carried out in the vacuum yields very good results. The reduction of the cæsium chloride by calcium is almost complete. It takes place with a certain violence, even when the cæsium or rubidium chloride and the calcium have been dried as carefully as possible. It is therefore necessary to apply the plug of asbestos behind the iron boat in order to keep back the mixture.

The frequencies of the line spectra of zinc and cadmium were determined in the following way: The line spectrum was photographed and

the photograph compared with the line spectrum on a transparent willemite screen. The characteristic lines on the willemite screen could be identified with the lines on the photographic plate without difficulty. Only those lines, which appeared on the willemite screen were allowed to act on the metal.

The strongest lines on the photographic plates are indicated by the following table.

Cd.	Zn.	Cu.
3615	3345	2769
2748	3302	2766
2573	2771	2369
2329	2756	2303
2321	2558	2294
2313	2502	2276
2288	2138	2242
2265	2064	2199
2144	2025	2149

Only those lines which produce phosphorescence on the willemite screen were allowed to affect the metal. The results of these measurements are given in the following tables, where l = wave-length, d = deflection, n = frequency, v = volts.

Cd Light.

l	$n \times 10^{-14}$	$n^2 \times 10^{-28}$	Na—K		Cæsium.	
			d	v	d	v
533.8	5.62	31.58	10.0	0.31	26.0	0.945
441.6	6.793	46.15	15.23	0.473	39.0	1.38
274.8	10.917	119.22	62.80	1.942	98.2	3.57
257.3	11.660	135.9	73.94	2.290	111.7	4.06
232.9	12.881	165.9	93.10	2.883	136.4	4.96
214.4	13.993	195.8	114.2	3.540	161.3	5.85

Zinc Light.

l	n	n^2	Na—K		Cæsium.	
			d	v	d	v
491.2	6.102	37.25	9.5	0.315	31.4	1.035
227.1	10.84	117.8	57.5	1.91	98.9	3.27
255.8	11.74	138.01	69.8	2.32	116.5	3.84
250.2	11.98	143.81	72.7	2.410	121.3	3.99
213.8	14.05	197.7	105.0	3.485	166.3	5.48
206.4	14.59	212.8	115.8	3.840	180.6	5.92
202.5	14.83	220.1	125.3	4.140	182.0	5.98

The positive potential increases continuously with the frequency, and the geometrical representation of these figures shows that it increases proportionally to the square of the frequency.

It has been found very difficult to prepare the metal with the same photoelectric properties twice in succession. The surface conditions play a very important part in the photoelectric effect; it is, therefore, hardly possible to draw definite conclusions as to the constants of the phenomena. When two samples of cæsium had been prepared under as nearly the same conditions as possible, the positive potentials were quite different for the same light; it might even happen that it was hardly active, and yet the surface appeared perfectly clean.

Though the positive potential increases as the square of the frequency, it was very noticeable that the parabola $v - n^2$ approaches in certain cases the straight line $v - n$, which we should expect from Planck's theory of radiation.

Towards the red end of the spectrum the photoelectric phenomena are different as is seen from the following numbers.

l	d	l	d
520	65.0	640	70.3
540	54.1	660	61.1
560	50.4	680	44.2
580	56.2	700	19.0
600	69.1	720	0.
620	74.2		

l = wave-length, d = deflection.

There is a maximum in the positive potential of sodium potassium alloy at $l = 618$. The positive potential in this region of the spectrum depends on the intensity of the light. Thus the positive potentials have to be expressed as function of the incident energy of radiation. This will be done in the following paper. In these experiments a Nernst glower was used as source of light while for the violet end of the spectrum an ordinary arc was sufficient.

Up to the present time all measurements indicate that in the violet and ultra-violet region of the spectrum the positive potential in the photoelectric effect is proportional to the square of the frequency and as Roentgen rays may be considered as electromagnetic disturbances of a certain thickness, and as they produce electrons of a high velocity, when incident on a metal plate, we have an experimental basis for determining the thickness of the Roentgen pulse or the wave-length equivalent, as it might conveniently be called, because the pulse in the ordinary primary Roentgen ray has very likely a certain structure, and in the secondary homogeneous Roentgen radiation there may be a certain amount of regular damped oscillation in the pulse. The present method does not

allow us to enter into these details of structure, but gives a means of determining from the photoelectric effect the resultant wave-length of the ordinary radiation which would be equivalent to the thickness of the Roentgen pulse.

It has been attempted to measure directly the positive potential excited by Roentgen rays instead of ultra-violet light. But the number of electrons emitted by Roentgen rays is so very small that the potential of the electrode rose very slowly and never reached a definite maximum. It was therefore necessary to measure the maximum velocity of the electrons emitted, rather than the positive potential in the case of equilibrium. The apparatus applied in these experiments is indicated by the following figure.

A brass tube is connected to the vacuum pump so that no light can enter the tube. The Roentgen rays penetrate through a hard-rubber tube A, 1.5 mm. thick, into the vacuum where they strike on a plate of zinc B. A part of the electrons will pass through two slits, one vertically above the other and make an impression on a photographic plate D so that C, D, E are on a straight line, when the magnetic field is not excited. When the tube however is placed between the poles of an electromagnet, the electrons will be deflected and make an impression at F. Knowing the distances CDE and EF and the magnetic field, it is easy to calculate the velocity of the electrons. The hardness of the Roentgen tube is adjusted so that it corresponds to a spark distance of 13 cm. It is necessary to introduce through the palladium from time to time a little hydrogen, in order to maintain the same hardness of the primary beam. The experiment is a very tedious one as it requires over ten hours exposure. The Roentgen tube gets so hot after a short time that the rays can only be produced intermittently; for instance, for five minutes, followed by three minutes rest and so on. As photographic plates I used Seed orthochromatic, a mercury turbine interrupter, and a large Klingelfuss induction coil. The magnetic field was measured by means of a magnetic balance. The photographic plate was cut into strips 5×1 cm. For a zinc plate the results of measurements were as follows:

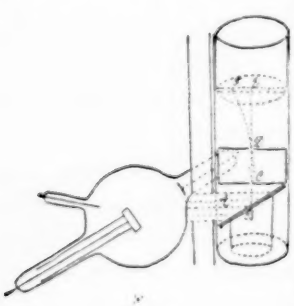

Fig. 2.

$$DC = 2 \text{ cm.} \quad DE = 5 \text{ cm.} \quad EF = 1.40 \text{ cm.}$$

The radius of curvature of the path of the electrons in the magnetic

field was equal to:

$r = 13.2$ cm., the magnetic field $H = 28.2$ absolute units.

The value $EF = 1.40$ cm. corresponds to the minimum deflection, or to the maximum velocity of the electrons. The line F on the photographic plate is drawn out somewhat which means that electrons of different velocities are given off. To the positive potential of equilibrium, measured in the previous experiments, corresponds the highest velocity of the electrons or the minimum deflection EF. This velocity is calculated by means of the formula:

$$mv^2/r = Hev, \quad e/m = 1.77. \times 10^7, \quad v = 6.6 \times 10^9, \quad H = 28.3, \quad r = 13.2.$$

This value corresponds well with those determined by P. D. Innes.[1]

For the same plate the positive potential acquired under the action of ultra-violet light was measured and found to be 1.88×10^8 absolute units for $l = 206.4$ mm., $P_2 = 1.88 \times 10^8$, $l_2 = 206.4$, $\mu\mu = 2.064 \times 10^{-5}$ cm. The potential P, corresponding to the velocity $v_1 = 6.6 \times 10^9$, is given by the formula:

$$\tfrac{1}{2}mv_1^2 = P_1 e, \quad P_1 = 1.239 \times 10^{12}.$$

Now we find the wave-length equivalent l_1 of Roentgen rays by the following equations, assuming that the curve passes through the origin, which is not correct:

$$P_1 : P_2 = n_1^2 : n_2^2, \quad n_1/n_2 = l_2/l_1 = \sqrt{P_1/P_2},$$
$$L_1 = 2.65 \times 10^{-7} \text{ cm.}$$

This would be the wave-length equivalent of those Roentgen rays which have been studied. The value determined by this principle of the photoelectric effect is large compared with those which have been estimated from other phenomena. The determination has been repeated with a lead plate instead of zinc, and with nearly the same result.

The measurements will be extended to other metals. It is hoped also that the principle here applied may be used for the determination of the wave-length equivalent of γ-rays and of secondary Roentgen radiation. The wave-length equivalent and the thickness of the pulse of Roentgen rays according to the electromagnetic theory must depend on the velocity of the primary cathode particles, and the study of the relations between the velocities of the primary and secondary electrons will throw some light on the forces that come into play in the collision between an electron and an atom.

[1] P. D. Innes, On the Velocity of the Cathode Particles, Proceedings of the Royal Society of London, 79, p. 442, 1907.

The wave-length equivalent l of Roentgen rays is equal to ct, where c is the velocity of light and t the time of collision between an electron and an atom. In the electron theory of radiation we assume, that the time of collision t is inversely proportional to the square of the velocity of the electron, $t = c_1/v^2$. If the radiation of the black body and the Roentgen rays are intimately related phenomena, then we conclude that:

$$\tfrac{1}{2}\, mv^2 = Pe; \quad t = c_3/P; \quad l = c_4/P.$$

Thus the wave-length equivalent of Roentgen rays would be inversely proportional to the potential difference between the cathode and anode. This last relation is susceptible of an experimental verification.

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URBANA, ILL.,
June 16, 1911.

NOTE ON A VARIABLE LOW RESISTANCE.

By J. H. Dellinger.

THERE is at present a tendency in electrical measurements to increase the precision of measurement of low resistances, inductances and electromotive forces by the use of low resistances in the bridge arms or in the potentiometer circuit. Such work requires a convenient low resistance, of small total resistance, capable of definite variation in practical cases of steps as small as one millionth of an ohm, which introduces no irregular variable resistance. Observers are so frequently troubled by the ordinary variable contacts that it seems worth while to point out the advantages of a simple device which anyone may construct for himself, viz., an amalgamated copper wire plunging in a tube of mercury. It is not known that a description of this device has been published, although the idea has been used in various forms by many experimenters. In the device here described the copper wire is used to short circuit the mercury, the operation depending on the fact that mercury has approximately 60 times the resistivity of copper. This principle has a notable advantage in alternating current work, which will be explained below.

The apparatus consists simply of a tube of thin glass (thick glass may be broken by scratching of the copper wire), or of porcelain or hard rubber, with a slight enlargement at the upper end to receive the mercury when displaced by the descending copper wire. An amalgamated copper wire, of slightly smaller diameter than the internal diameter of the tube, is lightly gripped either by a cork placed in the top of the tube, or by a pair of springs faced with insulating material. A length of 15 cm. is convenient. A 1-ohm range is obtained by using a tube of about 0.4 mm. diameter, and any other range down to 0.01 ohm. is obtained by tubes up to 4 mm. diameter. In the first of these mercury variable resistances, a flexible lead wire was connected to the upper end of the copper wire, and to prevent opening the circuit when the wire was wholly withdrawn from the mercury a wire shunt was connected across the terminals. An improvement in the connections was suggested by Dr. F. Wenner, viz., to connect simply to the top and bottom of the mercury column. Thus there is nothing whatever connected to the plunging copper wire, which functions simply by shortcircuiting the mercury. A

simple and valuable modification is the use of two of these resistances in series by the employment of a glass U-tube of which the diameter of one leg is $\sqrt{10}$ times the other. Thus a pair of variables of the ranges 0.1 ohm. and 0.01 ohm are conveniently obtained. The use of the U-tube eliminates the difficulty of sealing the lower end. A scale may readily be attached, and when the instrument has been calibrated the resistances of its settings are known. The instrument should not usually be used for exact quantitative work, however, as it has a large temperature coefficient.

The mercury variable resistance has been applied to facilitate adjusting to a balance the auxiliary ratio arms of the Thomson bridge, in the comparison of low resistance standards, and its use materially increases the speed of working. It is especially valuable in the measurement of the resistance of wire samples for determination of the conductivity, since the magnitude of the ratio arms may be very different for every sample measured. It has also been found to be a convenience in the comparison of mercury ohms with wire standard ohms, since the potential leads of the mercury standards are of relatively high resistance with respect to the ratio arms. This apparatus has also been usefully applied to the regulation of potentiometer currents.

The application of this device to alternating current bridge work has been productive of good results. The precision of determination of low inductances may be increased by the use of ratio arms of lower resistance. This has been made convenient by the use of the mercury variable resistance. It was found that, fortunately for alternating current work, this variable resistance has practically non-variable inductance (at any rate for moderate frequencies). This is evident, if the copper wire is of practically the same diameter as the mercury column, as the geometry of the circuit is not changed by varying the resistance. Thus this instrument is a complementary apparatus to the usual variable inductance. The total change of inductance of one of these "mercury variables" 15 cm. long, about 1.5 mm. internal diameter of tube, 1 mm. diameter of copper wire, and a range of 0.05 ohm, was $6\ (10)^{-9}$ henry, or 6 centimeters.

Bureau of Standards,
Washington, April 28, 1911.